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$U_q(sl(m+1))$-Module Algebra Structures on the Coordinate Algebra of a Quantum Vector Space

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Abstract. In this paper, the module-algebra structures of $U_q(sl(m+1)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{1\le i\le m}$ on the coordinate algebra of quantum vector spaces are studied. We denote the coordinate algebra of quantum n-dimensional vector space by $A_q(n)$. As our main result, first, we give a complete classification of module-algebra structures of $U_q(sl(m+1))$ on $A_q(3)$ when $k_i \in \mathrm{Aut}\ \mathrm{L}(A_q(3))$ as actions on $A_q(3)$ for $i = 1, \cdots, m$ and $m \ge 2$ and with the same method, on $A_q(2)$, all module-algebra structures of $U_q(sl(m+1))$ are characterized. Lastly, the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$ are obtained for any $n \ge 4$.
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1. Introduction

Quantum groups were introduced independently by Drinfeld in [7] and Jimbo in [16] which opened the floodgates for applications of Hopf algebras to physics, invariant theory for knots and links, and representations closely connected to Lie theory. Some basic references for quantum groups are [20] and [12]. Moreover, the actions of Hopf algebras [21] and their generalizations (see, e.g., [6]) play an important role in quantum group theory [18, 19] and in its various applications in physics [4]. However, it was long believed that the quantum plane [20] admits only one special symmetry [22] inspired by the action of $U_q(sl(2))$ (in other words the $U_q(sl(2))$-module algebra structure [18]). In [15], the coordinate algebra of quantum n-dimensional vector space is equipped with a special $U_q(sl(m+1))$-module algebra structure via a certain q-differential operator realization. Then it was shown [8], that the $U_q(sl(2))$-module algebra structure on the quantum plane

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is much richer and consists of 8 nonisomorphic cases [8, 9]. The full classification was given in terms of a so-called weight which was introduced for this purpose. Its introduction follows from the general form of an automorphism of the quantum plane [1]. Some properties of the actions of commutative Hopf algebras on quantum polynomials were studied in [2, 3]. In addition, there are also many papers studying Hopf algebras acting on some special algebras, for example, commutative domains, fields, filtered regular algebras and so on, see [5], [10] and [11].

Following [12], we consider here the actions of the quantum universal enveloping algebra $U_q(sl(m+1))$ on the coordinate algebra of quantum n-dimensional vector space $A_q(n)$. We use the method of weights [8, 9] to classify some actions in terms of action matrices which are introduced. We then present the Dynkin diagrams for the actions thus obtained and find their classical limit. A special case discussed in this paper was included in [15].

This work is organized as follows. In Section 2, we give the necessary preliminary information and notation, as well as proving an important lemma about actions on generators and any elements of $A_q(n)$. In Section 3, we study $U_q(sl(2))$-module algebra structures on $A_q(n)$ using the method of weights [8, 9]. The 0-th homogeneous component and 1-st homogeneous component of the action matrix are presented. In Section 4, we study the concrete actions of $U_q(sl(2))$ on $A_q(3)$ and characterize all module algebra structures of $U_q(sl(3))$ on $A_q(3)$ which make some preparations on the classification of module algebra structures of $U_q(sl(m+1))$ on $A_q(3)$. In Section 5, with the results of Section 4, all module-algebra structures of $U_q(sl(m+1))$ on $A_q(3)$ when $m \geq 2$ are presented. And, with the same method, all module-algebra structures of $U_q(sl(m+1))$ on $A_q(2)$ are given. Section 6 is devoted to study the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$ for $n \geq 4$.

In this paper, all algebras, modules and vector spaces are over the field $\mathbb{C}$ of complex numbers.

2. Preliminaries

Let H be a Hopf algebra whose comultiplication is Δ, counit is ε and antipode is S and let A be a unital algebra with unit 1. We use the Sweedler notation, such that $\Delta(h) = \sum_i h_i^{'} \otimes h_i^{''}$.

Definition 2.1. By a structure of an *H-module algebra* on A, we mean a homomorphism $\pi : H \to End_{\mathbb{C}} A$ such that:
(1) $\pi(h)(ab) = \sum_i \pi(h_i^{'})(a)\pi(h_i^{''})(b)$ for all $h \in H$, a, $b \in A$,
(2) $\pi(h)(1) = \varepsilon(h)1$ for all $h \in H$.

The structures π_1, π_2 are said to be *isomorphic*, if there exists an automorphism ψ of A such that $\psi\pi_1(h)\psi^{-1} = \pi_2(h)$ for all $h \in H$.

Throughout this paper, we assume that $q \in \mathbb{C}\backslash\{0\}$ and q is not a root of unity. We use the q-integers which were introduced by Heine [14] and are called the Heine numbers or q-deformed numbers [17] (for any integer $n > 0$)

$$(n)_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

First, we will introduce the definition of $U_q(sl(m+1))$.

Definition 2.2. The quantum universal enveloping algebra $U_q(sl(m+1))$ $(m \geq 1)$ as the algebra is generated by $(e_i, f_i, k_i, k_i^{-1})_{1 \leq i \leq m}$ with the relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \tag{2.1}$$

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \tag{2.2}$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \tag{2.3}$$

$$e_i e_j = e_j e_i \qquad \text{and} \qquad f_i f_j = f_j f_i, \qquad \text{if} \ \ a_{ij} = 0, \tag{2.4}$$

if $a_{ij} = -1$,

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0, \tag{2.5}$$

$$f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0, \tag{2.6}$$

where for any $i,\ j \in \{1, 2, \cdots, m\}$, $a_{ii} = 2$ and $a_{ij} = 0$, if $|i - j| > 1$; $a_{ij} = -1$, if $|i - j| = 1$.

The standard Hopf algebra structure on $U_q(sl(m+1))$ is determined by

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i, \tag{2.7}$$

$$\Delta(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1, \tag{2.8}$$

$$\Delta(k_i) = k_i \otimes k_i, \qquad \Delta(k_i^{-1}) = k_i^{-1} \otimes k_i^{-1}, \tag{2.9}$$

$$\varepsilon(k_i) = \varepsilon(k_i^{-1}) = 1, \tag{2.10}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \tag{2.11}$$

$$S(e_i) = -e_i k_i^{-1}, \qquad S(f_i) = -k_i f_i, \tag{2.12}$$

$$S(k_i) = k_i^{-1}, \qquad S(k_i^{-1}) = k_i, \tag{2.13}$$

for $i \in \{1, 2, \cdots, m\}$. We will use the notation $U_q(sl(m+1)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{1 \leq i \leq m}$.

Let us introduce the definition of the coordinate algebra of quantum n-dimensional vector space (see [13, 2]).

Definition 2.3. The *coordinate algebra of quantum n-dimensional vector space*, denoted by $A_q(n)$, is a unital algebra generated by n generators x_i for $i \in \{1, \cdots, n\}$ satisfying the relations

$$x_i x_j = q x_j x_i \quad \text{for} \quad \text{any} \quad i > j. \tag{2.14}$$

The coordinate algebra of quantum n-dimensional vector space $A_q(n)$ is also called a *quantum n-space*. If $n = 2$, $A_q(2)$ is also called a *quantum plane* (see [18]). In this case, Duplij and Sinel'shchikov studied the classification of $U_q(sl(2))$-module algebra structures on $A_q(2)$ in [8].

Next, we consider the automorphisms of $A_q(n)$. Obviously, $\varphi : A_q(n) \to A_q(n)$ is an automorphism defined as follows:

$$\varphi : \quad x_i \to \alpha_i x_i,$$

with $\alpha_i \in \mathbb{C} \setminus \{0\}$ for $i \in \{1, \cdots, n\}$. In addition, all such automorphisms form a subgroup of the automorphism group of $A_q(n)$. We denote this subgroup by Aut L$(A_q(n))$. It should be pointed out that there are other automorphisms of $A_q(3)$. For example, $\sigma : A_q(3) \to A_q(3)$ given by

$$\sigma(x_1) = \alpha_1 x_1, \quad \sigma(x_2) = \alpha_2 x_2 + \beta x_1 x_3, \quad \sigma(x_3) = \alpha_3 x_3,$$

with β and $\alpha_i \in \mathbb{C}\setminus\{0\}$ for $i \in \{1, 2, 3\}$ is an automorphism of $A_q(3)$. Throughout this paper, we restrict the actions of k_i in $U_q(sl(m+1))$ on $A_q(n)$ to Aut L$(A_q(n))$ for all m and n.

Finally, we present a lemma which will be useful for checking the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$.

Lemma 2.4. *Given the module-algebra actions of the generators k_i, e_i, f_i of $U_q(sl(m+1))$ on $A_q(n)$ for $i \in \{1, \cdots, m\}$, if an element in the ideal formed from the relations (2.1)-(2.6) of $U_q(sl(m+1))$ acting on the generators x_i of $A_q(n)$ produces zero for $i \in \{1, \cdots, n\}$, then this element acting on any $v \in A_q(n)$ produces zero.*

Proof. Here, we only prove that, if

$$e_i^2 e_{i+1}(x) - (q+q^{-1}) e_i e_{i+1} e_i(x) + e_{i+1} e_i^2(x) = 0 \text{ and}$$
$$e_i^2 e_{i+1}(y) - (q+q^{-1}) e_i e_{i+1} e_i(y) + e_{i+1} e_i^2(y) = 0, \text{ then}$$
$$e_i^2 e_{i+1}(xy) - (q+q^{-1}) e_i e_{i+1} e_i(xy) + e_{i+1} e_i^2(xy) = 0 \text{ where}$$

x, y are both generators of $A_q(n)$. The other relations can be proved similarly.
$e_i^2 e_{i+1}(xy) - (q+q^{-1}) e_i e_{i+1} e_i(xy) + e_{i+1} e_i^2(xy)$

$$\begin{aligned}
&= e_i^2(x e_{i+1}(y) + e_{i+1}(x) k_{i+1}(y)) - (q+q^{-1}) e_i e_{i+1}(x e_i(y) + e_i(x) k_i(y)) \\
&\quad + e_{i+1} e_i (x e_i(y) + e_i(x) k_i(y)) \\
&= e_i(x e_i e_{i+1}(y) + e_i(x) k_i e_{i+1}(y) + e_{i+1}(x) e_i k_{i+1}(y) + e_i e_{i+1}(x) k_i k_{i+1}(y)) \\
&\quad - (q+q^{-1}) e_i (x e_{i+1} e_i(y) + e_{i+1}(x) k_{i+1} e_i(y) + e_i(x) e_{i+1} k_i(y) + e_{i+1} e_i(x) \\
&\quad \cdot k_{i+1} k_i(y)) + e_{i+1}(x e_i^2(y) + e_i(x) k_i e_i(y) + e_i(x) e_i k_i(y) + e_i^2(x) k_i^2(y)) \\
&= (x e_i^2 e_{i+1}(y) - (q+q^{-1}) x e_i e_{i+1} e_i(y) + x e_{i+1} e_i^2(y)) + (e_i^2 e_{i+1}(x) - (q+q^{-1}) \\
&\quad \cdot e_i e_j e_i(x) + e_j e_i^2(x)) k_i^2 k_{i+1}(y) + (e_i(x) k_i e_i e_{i+1}(y) + e_i(x) e_i k_i e_{i+1}(y)
\end{aligned}$$

$$
\begin{aligned}
&-(q+q^{-1})e_i(x)e_ie_{i+1}k_i(y)) + (e_i^2(x)k_i^2e_{i+1}(y) - (q+q^{-1})e_i^2(x)\\
&\cdot k_ie_{i+1}k_i(y) + e_i^2(x)e_{i+1}k_i^2(y)) + (e_{i+1}(x)e_i^2k_{i+1}(y) - (q+q^{-1})e_{i+1}(x)\\
&\cdot e_ik_{i+1}e_i(y) + e_{i+1}(x)k_{i+1}e_i^2(y)) + (e_ie_{i+1}(x)k_ie_ik_{i+1}(y) + e_ie_{i+1}(x)\\
&\cdot e_ik_ik_{i+1}(y) - (q+q^{-1})e_ie_{i+1}(x)k_ik_{i+1}e_i(y)) + (-(q+q^{-1})e_{i+1}e_i(x)\\
&\cdot e_ik_{i+1}k_i(y) + e_{i+1}e_i(x)k_{i+1}k_ie_i(y) + e_{i+1}e_i(x)k_{i+1}e_ik_i(y))\\
&+(-(q+q^{-1})e_i(x)k_ie_{i+1}e_i(y) + e_i(x)e_{i+1}k_ie_i(y) + e_i(x)e_{i+1}e_ik_i(y))\\
&= 0.
\end{aligned}
$$

Thus, the lemma holds. ■

Therefore, by Lemma 2.4, in checking whether the relations of $U_q(sl(m+1))$, acting on any $v \in A_q(n)$, produce zero, we only need to check whether they produce zero when acting on the generators $x_1, \cdots, x_n$.

3. Properties of $U_q(sl(2))$-module algebras on $A_q(n)$

In this section, let us assume that $U_q(sl(2))$ is generated by k, e, f. Then, we will study the module-algebra structures of $U_q(sl(2))$ on $A_q(n)$ when $k \in \text{Aut L}(A_q(n))$ and $n \geq 3$.

By the definition of module algebra, it is easy to see that any action of $U_q(sl(2))$ on $A_q(n)$ is determined by the following $3 \times n$ matrix with entries from $A_q(n)$:

$$
M \stackrel{def}{=} \begin{bmatrix} k(x_1) & k(x_2) & \cdots & k(x_n) \\ e(x_1) & e(x_2) & \cdots & e(x_n) \\ f(x_1) & f(x_2) & \cdots & f(x_n) \end{bmatrix}, \tag{3.15}
$$

which is called the *action matrix* (see [8]). Given a $U_q(sl(2))$-module algebra structure on $A_q(n)$, obviously, the action of k determines an automorphism of $A_q(n)$. Therefore, by the assumption $k \in \text{Aut L}(A_q(n))$, we can set

$$
M_k \stackrel{def}{=} \begin{bmatrix} k(x_1) & k(x_2) & \cdots & k(x_n) \end{bmatrix} = \begin{bmatrix} \alpha_1 x_1 & \alpha_2 x_2 & \cdots & \alpha_n x_n \end{bmatrix},
$$

where α_i for $i \in \{1, \cdots, n\}$ are non-zero complex numbers. So, every monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} \in A_q(n)$ is an eigenvector for k and the associated eigenvalue $\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_n^{m_n}$ is called the *weight* of this monomial, which will be written as $wt(x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}) = \alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_n^{m_n}$.

Set

$$
M_{ef} \stackrel{def}{=} \begin{bmatrix} e(x_1) & e(x_2) & \cdots & e(x_n) \\ f(x_1) & f(x_2) & \cdots & f(x_n) \end{bmatrix}. \tag{3.16}
$$

Then, we have

$$
wt(M_{ef}) \stackrel{def}{=} \begin{bmatrix} wt(e(x_1)) & wt(e(x_2)) & \cdots & wt(e(x_n)) \\ wt(f(x_1)) & wt(f(x_2)) & \cdots & wt(f(x_n)) \end{bmatrix}
$$

$$
\bowtie \begin{bmatrix} q^2\alpha_1 & q^2\alpha_2 & \cdots & q^2\alpha_n \\ q^{-2}\alpha_1 & q^{-2}\alpha_2 & \cdots & q^{-2}\alpha_n \end{bmatrix},
$$

where the relation $A = (a_{ij}) \bowtie B = (b_{ij})$ means that for every pair of indices i, j such that both a_{ij} and b_{ij} are nonzero, one has $a_{ij} = b_{ij}$.

In the following, we denote the i-th homogeneous component of M, whose elements are just the i-th homogeneous components of the corresponding entries of M, by $(M)_i$. Set

$$(M)_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix}_0 .$$

Then, we obtain

$$wt((M)_0) \bowtie \begin{bmatrix} 0 & 0 & \cdots & 0 \\ q^2\alpha_1 & q^2\alpha_2 & \cdots & q^2\alpha_n \\ q^{-2}\alpha_1 & q^{-2}\alpha_2 & \cdots & q^{-2}\alpha_n \end{bmatrix}_0 \tag{3.17}$$

$$\bowtie \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}_0 .$$

An application of e and f to (2.14) gives the following equalities

$$x_i e(x_j) - q\alpha_i e(x_j) x_i = q x_j e(x_i) - \alpha_j e(x_i) x_j \quad \text{for} \quad i > j, \tag{3.18}$$
$$f(x_i) x_j - q\alpha_j^{-1} x_j f(x_i) = q f(x_j) x_i - \alpha_i^{-1} x_i f(x_j) \quad \text{for} \quad i > j. \tag{3.19}$$

After projecting the equalities above to $(A_q(n))_1$, we obtain

$$a_j(1 - q\alpha_i)x_i = a_i(q - \alpha_j)x_j \quad \text{for} \quad i > j;$$
$$b_i(1 - q\alpha_j^{-1})x_j = b_j(q - \alpha_i^{-1})x_i \quad \text{for} \quad i > j.$$

Therefore, for $i > j$, we obtain

$$a_j \neq 0 \Rightarrow \alpha_i = q^{-1}, \qquad a_i \neq 0 \Rightarrow \alpha_j = q, \tag{3.20}$$
$$b_i \neq 0 \Rightarrow \alpha_j = q, \qquad b_j \neq 0 \Rightarrow \alpha_i = q^{-1}. \tag{3.21}$$

Then, we have for any $j \in \{1, \cdots, n\}$,

$$a_j \neq 0 \Rightarrow \alpha_i = q^{-1} \;\; \text{for} \;\; \forall \;\; i > j, \;\; \alpha_i = q \;\; \text{for} \;\; \forall \;\; i < j, \tag{3.22}$$
$$b_j \neq 0 \Rightarrow \alpha_i = q^{-1} \;\; \text{for} \;\; \forall \;\; i > j, \;\; \alpha_i = q \;\; \text{for} \;\; \forall \;\; i < j. \tag{3.23}$$

By (3.17) and using the above equalities, we get

$$a_j \neq 0 \Rightarrow \alpha_1 = q, \cdots, \alpha_{j-1} = q, \alpha_j = q^{-2}, \alpha_{j+1} = q^{-1}, \cdots, \alpha_n = q^{-1},$$
$$b_j \neq 0 \Rightarrow \alpha_1 = q, \cdots, \alpha_{j-1} = q, \alpha_j = q^{2}, \alpha_{j+1} = q^{-1}, \cdots, \alpha_n = q^{-1}.$$

So, there are $2n + 1$ cases for 0-th homogeneous component of the action matrix as follows: $a_j \neq 0$, $a_i = 0$ for $i \neq j$ and all $b_i = 0$ for any $j \in \{1, \cdots, n\}$; $b_j \neq 0$,

$b_i = 0$ for $i \neq j$ and all $a_i = 0$ for any $j \in \{1, \cdots, n\}$; $a_j = 0$ and $b_j = 0$ for any $j \in \{1, \cdots, n\}$.

For the 1-st homogeneous component, since $wt(e(x_i)) = q^2 wt(x_i) \neq wt(x_i)$, we have $(e(x_i))_1 = \sum_{s\neq i} c_{is} x_s$ for some $c_{is} \in \mathbb{C}$. Similarly, we set $(f(x_i))_1 = \sum_{s\neq i} d_{is} x_s$ for some $d_{is} \in \mathbb{C}$.

After projecting Equations (3.18)-(3.19) to $(A_q(n))_2$, we can obtain, for any $i > j$,

$$\sum_{\substack{s\neq j\\ s<i}} (q - q\alpha_i) c_{js} x_s x_i + (1 - q\alpha_i) c_{ji} x_i^2 + \sum_{s>i} (1 - q^2\alpha_i) c_{js} x_i x_s =$$
$$\sum_{s<j} (q^2 - \alpha_j) c_{is} x_s x_j + (q - \alpha_j) c_{ij} x_j^2 + \sum_{\substack{s\neq i\\ s>j}} (q - q\alpha_j) c_{is} x_j x_s,$$
$$\sum_{s<j} (1 - q^2\alpha_j^{-1}) d_{is} x_s x_j + (1 - q\alpha_j^{-1}) d_{ij} x_j^2 + \sum_{\substack{s>j\\ s\neq i}} (q - q\alpha_j^{-1}) d_{is} x_j x_s =$$
$$\sum_{\substack{s<i\\ s\neq j}} (q - q\alpha_i^{-1}) d_{js} x_s x_i + (q - \alpha_i^{-1}) d_{ji} x_i^2 + \sum_{s>i} (q^2 - \alpha_i^{-1}) d_{js} x_i x_s.$$

Therefore, we have

$$\begin{aligned}
&c_{js} \neq 0 \ \ (s < i, \ \ s \neq j) \Rightarrow \alpha_i = 1, \ \ c_{js} = 0 \ \text{ for all } \ s \geq i,\\
&c_{ji} \neq 0 \Rightarrow \alpha_i = q^{-1}, \ \ c_{js} = 0 \ \text{ for any } \ s \neq i,\\
&c_{js} \neq 0 \ \ (s > i) \Rightarrow \alpha_i = q^{-2}, \ \ c_{js} = 0 \ \text{ for all } \ s \leq i,\\
&c_{is} \neq 0 \ \ (s < j) \Rightarrow \alpha_j = q^{2}, \ \ c_{is} = 0 \ \text{ for all } \ s \geq j,\\
&c_{ij} \neq 0 \Rightarrow \alpha_j = q, \ \ c_{is} = 0 \ \text{ for all } \ s \neq j,\\
&c_{is} \neq 0 \ \ (s > j, \ \ s \neq i) \Rightarrow \alpha_j = 1, \ \ c_{is} = 0 \ \text{ for all } \ s \leq j,\\
&d_{is} \neq 0 \ \ (s < j) \Rightarrow \alpha_j = q^{2}, \ \ d_{is} = 0 \ \text{ for all } \ s > j,\\
&d_{ij} \neq 0 \Rightarrow \alpha_j = q, \ \ d_{is} = 0 \ \text{ for all } \ s \neq j,\\
&d_{is} \neq 0 \ \ (s > j, \ \ s \neq i) \Rightarrow \alpha_j = 1, \ \ d_{is} = 0 \ \text{ for all } \ s \leq j,\\
&d_{js} \neq 0 \ \ (s < i, \ \ s \neq j) \Rightarrow \alpha_i = 1, \ \ d_{js} = 0 \ \text{ for all } \ s \geq i,\\
&d_{ji} \neq 0 \Rightarrow \alpha_i = q^{-1}, \ \ d_{js} = 0 \ \text{ for all } \ s \neq i,\\
&d_{js} \neq 0 \ \ (s > i) \Rightarrow \alpha_i = q^{-2}, \ \ d_{js} = 0 \ \text{ for all } \ s \leq i.
\end{aligned}$$

Therefore, we have for any $j \in \{1, \cdots, n\}$,

$$\begin{aligned}
c_{js} \neq 0 \ \ (s > j) \ \ &\Rightarrow \ \ \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_{j+1} = q^{-2}, \cdots,\\
&\qquad \alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1,\\
c_{js} \neq 0 \ \ (s < j) \ \ &\Rightarrow \ \ \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots,\\
&\qquad \alpha_{j-1} = q^2, \alpha_{j+1} = 1, \cdots, \alpha_n = 1,\\
d_{js} \neq 0 \ \ (s > j) \ \ &\Rightarrow \ \ \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_{j+1} = q^{-2}, \cdots,\\
&\qquad \alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1,\\
d_{js} \neq 0 \ \ (s < j) \ \ &\Rightarrow \ \ \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots,\\
&\qquad \alpha_{j-1} = q^2, \alpha_{j+1} = 1, \cdots, \alpha_n = 1.
\end{aligned}$$

Since $wt((M_{ef})_1) = \begin{bmatrix} q^2\alpha_1 & q^2\alpha_2 & \cdots & q^2\alpha_n \\ q^{-2}\alpha_1 & q^{-2}\alpha_2 & \cdots & q^{-2}\alpha_n \end{bmatrix}_1$, we obtain for any $j \in \{1, \cdots, n\}$,

$$\begin{aligned}
c_{js} \neq 0 \quad (s > j) \quad &\Rightarrow \quad \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_j = q^{-3}, \alpha_{j+1} = q^{-2}, \cdots, \\
&\qquad \alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1, \\
c_{js} \neq 0 \quad (s < j) \quad &\Rightarrow \quad \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots, \\
&\qquad \alpha_{j-1} = q^2, \alpha_j = q^{-1}, \alpha_{j+1} = 1, \cdots, \alpha_n = 1, \\
d_{js} \neq 0 \quad (s > j) \quad &\Rightarrow \quad \alpha_1 = 1, \cdots, \alpha_{j-1} = 1, \alpha_j = q, \alpha_{j+1} = q^{-2}, \cdots, \\
&\qquad \alpha_{s-1} = q^{-2}, \alpha_s = q^{-1}, \alpha_{s+1} = 1, \cdots, \alpha_n = 1, \\
d_{js} \neq 0 \quad (s < j) \quad &\Rightarrow \quad \alpha_1 = 1, \cdots, \alpha_{s-1} = 1, \alpha_s = q, \alpha_{s+1} = q^2, \cdots, \\
&\qquad \alpha_{j-1} = q^2, \alpha_j = q^3, \alpha_{j+1} = 1, \cdots, \alpha_n = 1.
\end{aligned}$$

By the above discussion, we have only the following possibilities for the 1-st homogeneous component: $c_{ij} \neq 0$ for some $i \neq j$, other c_{st} equal to zero and all $d_{st} = 0$; $d_{ij} \neq 0$ for some $i \neq j$, other d_{st} equal to zero and all $c_{st} = 0$; $c_{j+1,j} \neq 0$, $d_{j,j+1} \neq 0$ for some $j \in \{1, \cdots, n\}$.

Obviously, if both the 0-th homogeneous component and the 1-st homogeneous component of M_{ef} are nonzero, there are no possibilities except when $n = 3$. For $n = 3$, there are only two possibilities (a_2, d_{13}, b_2, $c_{31} \in \mathbb{C}\backslash\{0\}$):

$$\left(\begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ d_{13} & 0 & 0 \end{bmatrix}_1 \right) \Rightarrow \alpha_1 = q, \alpha_2 = q^{-2}, \alpha_3 = q^{-1}, \tag{3.24}$$

$$\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & b_2 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & c_{31} \\ 0 & 0 & 0 \end{bmatrix}_1 \right) \Rightarrow \alpha_1 = q, \alpha_2 = q^2, \alpha_3 = q^{-1}. \tag{3.25}$$

Moreover, there are no possibilities when the 0-th homogeneous component of M_{ef} is 0 and the 1-st homogeneous component of M_{ef} have only one nonzero position. The reasons are the same as those in [8].

Therefore, by the above discussion, we can obtain the following theorem.

Theorem 3.1. *Given a module algebra structure of $U_q(sl(2))$ on $A_q(n)$. The 0-th homogeneous component and 1-st homogeneous component of the action matrix must be one of the following cases:*
Case (3.24), *Case* (3.25) *when* $n = 3$,
$\left(\begin{bmatrix} 0 & 0 & \cdots & a_i & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_1 \right)$, $a_i \neq 0$ *for any* $i \in \{1, \cdots n\}$,
$\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & b_i & \cdots & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_1 \right)$, $b_i \neq 0$ *for any* $i \in \{1, \cdots n\}$,
$\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & \cdots & 0 & c_{j+1,j} & \cdots & 0 \\ 0 & 0 & \cdots & d_{j,j+1} & 0 & \cdots & 0 \end{bmatrix}_1 \right)$, $c_{j+1,j}$, $d_{j,j+1} \neq 0$ *for any* $j \in \{1, \cdots, n-1\}$,
$\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_1 \right)$ *for any* $n \geq 3$.

4. General structure of $U_q(sl(m+1))$-module algebras on $A_q(3)$

In this section, we study the concrete actions of $U_q(sl(2))$ on $A_q(3)$ and module algebra structures of $U_q(sl(3))$ on $A_q(3)$ which make some preparations on the classification of module algebra structures of $U_q(sl(m+1))$ on $A_q(3)$ for $m \geq 2$.

By Theorem 3.1, we only need to consider 11 possibilities
$\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1\right)$, $\left(\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1\right)$,
$\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_3 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1\right)$, $\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & c_{21} & 0 \\ d_{12} & 0 & 0 \end{bmatrix}_1\right)$,
$\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & c_{32} \\ 0 & d_{23} & 0 \end{bmatrix}_1\right)$, $\left(\begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1\right)$,
$\left(\begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ d_{13} & 0 & 0 \end{bmatrix}_1\right)$, $\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & b_2 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1\right)$,
$\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & b_2 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & c_{31} \\ 0 & 0 & 0 \end{bmatrix}_1\right)$, $\left(\begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1\right)$,
$\left(\begin{bmatrix} 0 & 0 & 0 \\ b_1 & 0 & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1\right)$ where $a_i \neq 0$, $b_i \neq 0$ for $i = 1, 2, 3$, and c_{21}, d_{12}, c_{32}, d_{23}, d_{13}, c_{31} are not zero.

For convenience, we denote these 11 kinds of cases in the above order by $(*_1)$, $\cdots$, $(*_{11})$ respectively.

Lemma 4.1. *For Case $(*_1)$, all $U_q(sl(2))$-module algebra structures on $A_q(3)$ are as follows*

$$k(x_1) = \pm x_1, \quad k(x_2) = \pm x_2, \quad k(x_3) = \pm x_3, \tag{4.26}$$
$$e(x_1) = e(x_2) = e(x_3) = f(x_1) = f(x_2) = f(x_3) = 0. \tag{4.27}$$

Proof. The proof is similar to that in Theorem 4.2 in [8]. ■

Lemma 4.2. *For Case $(*_2)$, all $U_q(sl(2))$-module algebra structures on $A_q(3)$ are*

$$k(x_1) = q^{-2}x_1, \quad k(x_2) = q^{-1}x_2, \quad k(x_3) = q^{-1}x_3, \tag{4.28}$$
$$e(x_1) = a_1, \quad e(x_2) = 0, \quad e(x_3) = 0, \tag{4.29}$$
$$f(x_1) = -qa_1^{-1}x_1^2, \tag{4.30}$$
$$f(x_2) = -qa_1^{-1}x_1x_2 + \xi_1 x_2 x_3^2 + \xi_2 x_2^3 + \xi_3 x_3^3, \tag{4.31}$$
$$f(x_3) = -qa_1^{-1}x_1x_3 + \xi_4 x_2 x_3^2 + (1+q+q^2)\xi_2 x_2^2 x_3 - q^{-1}\xi_1 x_3^3, \tag{4.32}$$

where $a_1 \in \mathbb{C}\backslash\{0\}$, and ξ_1, ξ_2, ξ_3, $\xi_4 \in \mathbb{C}$.

Proof. Since $wt(M_{ef}) \bowtie \begin{bmatrix} 1 & q & q \\ q^{-4} & q^{-3} & q^{-3} \end{bmatrix}$ and $\alpha_1 = q^{-2}$, $\alpha_2 = q^{-1}$, $\alpha_3 = q^{-1}$, we must have $e(x_1) = a_1$, $e(x_2) = 0$, $e(x_3) = 0$. For the same reason, $f(x_1)$, $f(x_2)$, $f(x_3)$ must be of the following forms:
$f(x_1) = u_1x_1^2 + u_2x_1x_2^2 + u_3x_1x_3^2 + u_4x_1x_2x_3 + u_5x_2x_3^3 + u_6x_2^2x_3^2 + u_7x_2^3x_3 + u_8x_2^4 + u_9x_3^4$,
$f(x_2) = v_1x_1x_2 + v_2x_1x_3 + v_3x_2x_3^2 + v_4x_2^2x_3 + v_5x_2^3 + v_6x_3^3$,

$f(x_3) = w_1x_1x_2 + w_2x_1x_3 + w_3x_2x_3^2 + w_4x_2^2x_3 + w_5x_2^3 + w_6x_3^3$,
where these coefficients are in $\mathbb{C}$. Then, we consider (3.18)-(3.19). Taking $e(x_1)$, $e(x_2)$, $e(x_3)$, $f(x_1)$, $f(x_2)$, $f(x_3)$ into the six equalities, and comparing the coefficients, we obtain

$$\begin{aligned}
&f(x_1) = u_1x_1^2,\\
&f(x_2) = u_1x_1x_2 + v_3x_2x_3^2 + v_5x_2^3 + v_6x_3^3,\\
&f(x_3) = u_1x_1x_3 + w_3x_2x_3^2 + (1+q+q^2)v_5x_2^2x_3 - q^{-1}v_3x_3^3.
\end{aligned}$$

Using $ef(u) - fe(u) = \frac{k-k^{-1}}{q-q^{-1}}(u)$, for any $u \in \{x_1, x_2, x_3\}$, we get $u_1 = -qa_1^{-1}$. So, the proof is finished. ∎

Lemma 4.3. *For Case* $(*_3)$, *all* $U_q(sl(2))$*-module algebra structures on* $A_q(3)$ *are as follows*

$$k(x_1) = qx_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^2x_3, \tag{4.33}$$
$$e(x_1) = -qb_3^{-1}x_1x_3 + \mu_1x_1^2x_2 - q\mu_2x_1^3 + (1+q+q^2)\mu_3x_1x_2^2, \tag{4.34}$$
$$e(x_2) = -qb_3^{-1}x_2x_3 + \mu_2x_1^2x_2 + \mu_3x_2^3 + \mu_4x_1^3, \tag{4.35}$$
$$e(x_3) = -qb_3^{-1}x_3^2, \tag{4.36}$$
$$f(x_1) = 0, \quad f(x_2) = 0, \quad f(x_3) = b_3, \tag{4.37}$$

where $b_3 \in \mathbb{C}\backslash\{0\}$, *and* μ_1, μ_2, μ_3 $\mu_4 \in \mathbb{C}$.

Proof. The proof is similar to that in Lemma 4.2. ∎

Lemma 4.4. *For Case* $(*_4)$, *to satisfy (3.18)-(3.19), the actions of* k, e, f *must be of the following form*

$$\begin{aligned}
&k(x_1) = qx_1, \quad k(x_2) = q^{-1}x_2, \quad k(x_3) = x_3,\\
&e(x_1) = \sum_{\substack{m\geq 0, p\geq 0\\ p\neq m+3}} a_{m,p}x_1^{m+3}x_2^mx_3^p + \sum_{m\geq 0} d_mx_1^{m+3}x_2^mx_3^{m+3},\\
&e(x_2) = c_{21}x_1 + \sum_{\substack{m\geq 0, p\geq 0\\ p\neq m+3}} b_{m,p}x_1^{m+2}x_2^{m+1}x_3^p,\\
&e(x_3) = \sum_{\substack{m\geq 0, p\geq 0\\ p\neq m+3}} \overline{c_{m,p}}x_1^{m+2}x_2^mx_3^{p+1} + \sum_{m\geq 0} e_mx_1^{m+2}x_2^mx_3^{m+4},\\
&f(x_1) = d_{12}x_2 + \sum_{\substack{m\geq 0, p\geq 0\\ p\neq m+1}} \overline{d_{m,p}}x_1^{m+1}x_2^{m+2}x_3^p + \sum_{m\geq 0} h_mx_1^{m+1}x_2^{m+2}x_3^{m+1},\\
&f(x_2) = \sum_{\substack{m\geq 0, p\geq 0\\ p\neq m+1}} e_{m,p}x_1^mx_2^{m+3}x_3^p,\\
&f(x_3) = \sum_{\substack{m\geq 0, p\geq 0\\ p\neq m+1}} g_{m,p}x_1^mx_2^{m+2}x_3^{p+1} + \sum_{m\geq 0} g_mx_1^mx_2^{m+2}x_3^{m+2},
\end{aligned}$$

where $c_{21}, d_{12} \in \mathbb{C}\backslash\{0\}$, *the other coefficients are in* $\mathbb{C}$ *and* $\frac{a_{m,p}}{b_{m,p}} = -\frac{(m+p+1)_q}{q^{p-1}(m+3-p)_q}$, $\frac{b_{m,p}}{\overline{c_{m,p}}} = \frac{q^{p-1}(m+3-p)_q}{(2m+2)_q}$, $\frac{d_m}{e_m} = -\frac{(2m+4)_q}{(2m+2)_q}$, $\frac{\overline{d_{m,p}}}{e_{m,p}} = -\frac{(m+p+3)_q}{q^{p+1}(m+1-p)_q}$, $\frac{\overline{d_{m,p}}}{g_{m,p}} = -\frac{(m+p+3)_q}{q(2m+2)_q}$, $\frac{h_m}{g_m} = -\frac{(2m+4)_q}{q(2m+2)_q}$.

In particular, there are the following $U_q(sl(2))$ *-module algebra structures on* $A_q(3)$

$$k(x_1) = qx_1, \quad k(x_2) = q^{-1}x_2, \quad k(x_3) = x_3, \tag{4.38}$$

$$e(x_1) = 0, \quad e(x_2) = c_{21}x_1, \quad e(x_3) = 0, \tag{4.39}$$

$$f(x_1) = c_{21}^{-1}x_2, \quad f(x_2) = 0, \quad f(x_3) = 0, \tag{4.40}$$

where $c_{21} \in \mathbb{C}\backslash\{0\}$.

Proof. In this case, we get $\alpha_1 = q$, $\alpha_2 = q^{-1}$, $\alpha_3 = 1$. Therefore, we obtain $wt(M_{ef}) \bowtie \begin{bmatrix} q^3 & q & q^2 \\ q^{-1} & q^{-3} & q^{-2} \end{bmatrix}$. Since $wt(e(x_1)) = q^3$, $wt(e(x_2)) = q$ and $wt(e(x_3)) = q^2$, using (3.18) and by some computations, we can obtain $e(x_1)$, $e(x_2)$ and $e(x_3)$ in the forms appearing in the lemma. Similarly, we also can determine the forms of $f(x_1)$, $f(x_2)$ and $f(x_3)$ in the lemma.

Note that (4.38)-(4.40) determine the module-algebra structures of $U_q(sl(2))$ on $A_q(3)$. ∎

Lemma 4.5. *For Case* $(*_5)$, *to satisfy (3.18)-(3.19), the actions of* $U_q(sl(2))$ *on* $A_q(3)$ *must be of the following form*

$$k(x_1) = x_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-1}x_3,$$

$$e(x_1) = \sum_{\substack{m\geq 0, p\geq 0 \\ p\neq m+1}} \widetilde{a_{m,p}} x_1^{p+1} x_2^{2+m} x_3^m + \sum_{m\geq 0} \widetilde{a_m} x_1^{m+2} x_2^{m+2} x_3^m,$$

$$e(x_2) = \sum_{\substack{m\geq 0, p\geq 0 \\ p\neq m+1}} \widetilde{b_{m,p}} x_1^p x_2^{3+m} x_3^m,$$

$$e(x_3) = c_{32}x_2 + \sum_{\substack{m\geq 0, p\geq 0 \\ p\neq m+1}} \widetilde{c_{m,p}} x_1^p x_2^{m+2} x_3^{m+1} + \sum_{m\geq 0} \widetilde{c_m} x_1^{m+1} x_2^{m+2} x_3^{m+1},$$

$$f(x_1) = \sum_{\substack{m\geq 0, p\geq 0 \\ p\neq m+3}} \widetilde{d_{m,p}} x_1^{p+1} x_2^m x_3^{m+2} + \sum_{m\geq 0} \widetilde{d_m} x_1^{m+4} x_2^m x_3^{m+2},$$

$$f(x_2) = d_{23}x_3 + \sum_{\substack{m\geq 0, p\geq 0 \\ p\neq m+3}} \widetilde{e_{m,p}} x_1^p x_2^{m+1} x_3^{m+2},$$

$$f(x_3) = \sum_{\substack{m\geq 0, p\geq 0 \\ p\neq m+3}} \widetilde{g_{m,p}} x_1^p x_2^m x_3^{m+3} + \sum_{m\geq 0} \widetilde{g_m} x_1^{m+3} x_2^m x_3^{m+3},$$

where c_{32}, $d_{23} \in \mathbb{C}\backslash\{0\}$, *other coefficients are in* $\mathbb{C}$ *and* $\frac{\widetilde{a_{m,p}}}{\widetilde{b_{m,p}}} = \frac{(2m+2)_q}{q^p(m-p+1)_q}$, $\frac{\widetilde{a_{m,p}}}{\widetilde{c_{m,p}}} = -\frac{q(2m+2)_q}{(m+p+3)_q}$, $\frac{\widetilde{a_m}}{\widetilde{c_m}} = -\frac{q(2m+2)_q}{(2m+4)_q}$, $\frac{\widetilde{d_{m,p}}}{\widetilde{e_{m,p}}} = \frac{(2m+2)_q}{q^{p-1}(m-p+3)_q}$, $\frac{\widetilde{d_{m,p}}}{\widetilde{g_{m,p}}} = -\frac{(2m+2)_q}{(m+p+1)_q}$, $\frac{\widetilde{d_m}}{\widetilde{g_m}} = -\frac{(2m+2)_q}{(2m+4)_q}$.

There are the following $U_q(sl(2))$-module algebra structures on $A_q(3)$

$$k(x_1) = x_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-1}x_3, \tag{4.41}$$
$$e(x_1) = 0, \quad e(x_2) = 0, \quad e(x_3) = c_{32}x_2, \tag{4.42}$$
$$f(x_1) = 0, \quad f(x_2) = c_{32}^{-1}x_3, \quad f(x_3) = 0, \tag{4.43}$$

where $c_{32} \in \mathbb{C}\backslash\{0\}$.

Proof. The proof is similar to that in Lemma 4.4. ■

Lemma 4.6. *For Case $(*_6)$ and Case $(*_7)$, to satisfy (3.18)-(3.19), the actions of k, e and f on $A_q(3)$ are*

$$\begin{aligned}
k(x_1) &= qx_1, \quad k(x_2) = q^{-2}x_2, \quad k(x_3) = q^{-1}x_3, \\
e(x_1) &= \sigma x_1^3 + \sum_{n\geq 0} \sigma_n x_1^{2n+5}x_2^{n+1} + \sum_{p\geq 0} \widetilde{\sigma}_p x_1^{p+4}x_3^{p+1} \\
&\quad + \sum_{n\geq 0, p\geq 0} \sigma_{n,p} x_1^{2n+p+6}x_2^{n+1}x_3^{p+1}, \\
e(x_2) &= a_2 + \rho x_1^2 x_2 + \sum_{n\geq 0} \rho_n x_1^{2n+4}x_2^{n+2} + \sum_{p\geq 0} \widehat{\rho}_p x_1^{p+3}x_2 x_3^{p+1} \\
&\quad + \sum_{n\geq 0, p\geq 0} \rho_{n,p} x_1^{2n+p+5}x_2^{n+2}x_3^{p+1}, \\
e(x_3) &= \tau x_1^2 x_3 + \sum_{n\geq 0} \tau_n x_1^{2n+4}x_2^{n+1}x_3 + \sum_{p\geq 0} \widetilde{\tau}_p x_1^{p+3}x_3^{p+2} \\
&\quad + \sum_{n\geq 0, p\geq 0} \tau_{n,p} x_1^{2n+p+5}x_2^{n+1}x_3^{p+2}, \\
f(x_1) &= d_{13}x_3 + \sum_{p\geq 0} \lambda_p x_1^{p+1}x_3^{p+2} + \sum_{n\geq 0} \widetilde{\lambda_n} x_1^{2n+1}x_2^{n+1} + \sum_{n\geq 0} \widehat{\lambda_n} x_1^{2n+2}x_2^{n+1}x_3 \\
&\quad + \sum_{n\geq 0, p\geq 0} \lambda_{n,p} x_1^{2n+p+3}x_2^{n+1}x_3^{p+2}, \\
f(x_2) &= \sum_{n\geq 0} \widetilde{\nu_n} x_1^{2n}x_2^{n+2} + \sum_{n\geq 0} \widehat{\nu_n} x_1^{2n+1}x_2^{n+2}x_3 + \sum_{n\geq 0, p\geq 0} \nu_{n,p} x_1^{2n+p+2}x_2^{n+2}x_3^{p+2}, \\
f(x_3) &= \sum_{p\geq 0} \omega_p x_1^p x_3^{p+3} + \sum_{n\geq 0} \widetilde{\omega_n} x_1^{2n}x_2^{n+1}x_3 + \sum_{n\geq 0} \widehat{\omega_n} x_1^{2n+1}x_2^{n+1}x_3^2 \\
&\quad + \sum_{n\geq 0, p\geq 0} \omega_{n,p} x_1^{2n+p+2}x_2^{n+1}x_3^{p+3},
\end{aligned}$$

where $a_2 \in \mathbb{C}\backslash\{0\}$ and the other coefficients are in $\mathbb{C}$, and $\frac{\sigma}{\rho} = -\frac{q^2}{(4)_q}$, $\frac{\sigma_n}{\rho_n} = -\frac{q^2(n+2)_q}{(2n+6)_q}$, $\frac{\widetilde{\sigma_p}}{\widetilde{\rho_p}} = -\frac{(p+2)_q}{q^{p-1}(4)_q}$, $\frac{\sigma_{n,p}}{\rho_{n,p}} = -\frac{(n+p+3)_q}{q^{p-1}(2n+6)_q}$, $\frac{\sigma}{\tau} = -\frac{q}{(3)_q}$, $\frac{\sigma_n}{\tau_n} = -\frac{q(n+2)_q}{(3n+6)_q}$, $\frac{\widetilde{\sigma_p}}{\widetilde{\tau}_p} = -\frac{q(p+2)_q}{(p+4)_q}$, $\frac{\sigma_{n,p}}{\tau_{n,p}} = -\frac{q(n+p+3)_q}{(3n+p+7)_q}$, $\frac{\lambda_p}{\omega_p} = -\frac{(p+3)_q}{q(p+1)_q}$, $\frac{\widetilde{\lambda_n}}{\widetilde{\omega_n}} = -\frac{(n+2)_q}{q(3n+2)_q}$, $\frac{\widehat{\lambda_n}}{\widehat{\omega_n}} = -\frac{(n+3)_q}{q(3n+3)_q}$, $\frac{\lambda_{n,p}}{\omega_{n,p}} = -\frac{(n+p+4)_q}{q(3n+p+4)_q}$, $\frac{\widetilde{\lambda_n}}{\widetilde{\nu_n}} = -\frac{(n+2)_q}{q(2n+2)_q}$, $\frac{\widehat{\lambda_n}}{\widehat{\nu_n}} = -\frac{(n+3)_q}{q^2(2n+2)_q}$, $\frac{\lambda_{n,p}}{\nu_{n,p}} = -\frac{(n+p+4)_q}{q^{p+3}(2n+2)_q}$.

In particular, there are the following $U_q(sl(2))$-module algebra structures on

$A_q(3)$:

$$k(x_1) = qx_1, \quad k(x_2) = q^{-2}x_2, \quad k(x_3) = q^{-1}x_3, \tag{4.44}$$

$$e(x_1) = 0, \quad e(x_2) = a_2, \quad e(x_3) = 0, \tag{4.45}$$

$$f(x_1) = d_{13}x_3 + a_2^{-1}x_1x_2 + \sum_{p=0}^{n} \widehat{d_p} x_1^{p+1}x_3^{p+2}, \quad f(x_2) = -qa_2^{-1}x_2^2, \tag{4.46}$$

$$f(x_3) = -qa_2^{-1}x_2x_3 - \sum_{p=0}^{n} \frac{q(p+1)_q}{(p+3)_q} \widehat{d_p} x_1^p x_3^{p+3}, \tag{4.47}$$

where $n \in \mathbf{N}$, d_{13}, $\widehat{d_p} \in \mathbb{C}$ *for all* p, $a_2 \in \mathbb{C}\backslash\{0\}$.

Proof. In these two cases, we have the same values of α_1, α_2 and α_3, i.e., $\alpha_1 = q$, $\alpha_2 = q^{-2}$, $\alpha_3 = q^{-1}$. Therefore, $wt(M_{ef}) \bowtie \begin{bmatrix} q^3 & 1 & q \\ q^{-1} & q^{-4} & q^{-3} \end{bmatrix}$. Using the equalities (3.18)-(3.19) and by some computations, we can obtain that $e(x_1)$, $e(x_2)$, $e(x_3)$, $f(x_1)$, $f(x_2)$, $f(x_3)$ are of the forms in this lemma.

Moreover, using (4.44)-(4.47), it is easy to check that $ef(u) - fe(u) = \frac{k-k^{-1}}{q-q^{-1}}(u)$, where $u \in \{x_1, x_2, x_3\}$. Therefore, they determine the module-algebra structures of $U_q(sl(2))$ on $A_q(3)$. ∎

Lemma 4.7. *For Case* $(*_8)$ *and Case* $(*_9)$*, to satisfy (3.18)-(3.19), the actions of* k, e, f *are of the form*

$$\begin{aligned}
k(x_1) &= qx_1, \quad k(x_2) = q^2x_2, \quad k(x_3) = q^{-1}x_3, \\
e(x_1) &= \sum_{p\geq 0} \alpha_p x_1^{p+3}x_3^p + \sum_{m\geq 0} \widetilde{\alpha_m} x_1x_2^{m+1}x_3^{2m} + \sum_{m\geq 0} \widehat{\alpha_m} x_1^2x_2^{m+1}x_3^{2m+1} \\
&\quad + \sum_{p\geq 0, m\geq 0} \alpha_{m,p} x_1^{p+3}x_2^{m+1}x_3^{2m+p+2}, \\
e(x_2) &= \sum_{m\geq 0} \widetilde{\beta_m} x_2^{m+2}x_3^{2m} + \sum_{m\geq 0} \widehat{\beta_m} x_1x_2^{m+2}x_3^{2m+1} \\
&\quad + \sum_{p\geq 0, m\geq 0} \beta_{m,p} x_1^{p+2}x_2^{m+2}x_3^{2m+p+2}, \\
e(x_3) &= c_{31}x_1 + \sum_{p\geq 0} \gamma_p x_1^{p+2}x_3^{p+1} + \sum_{m\geq 0} \widetilde{\gamma_m} x_2^{m+1}x_3^{2m+1} + \sum_{m\geq 0} \widehat{\gamma_m} x_1x_2^{m+1}x_3^{2m+2} \\
&\quad + \sum_{p\geq 0, m\geq 0} \gamma_{m,p} x_1^{p+2}x_2^{m+1}x_3^{2m+p+3}, \\
f(x_1) &= \varepsilon x_1x_3^2 + \sum_{p\geq 0} \varepsilon_p x_1^{p+2}x_3^{p+3} + \sum_{m\geq 0} \widetilde{\varepsilon_m} x_1x_2^{m+1}x_3^{2m+4} \\
&\quad + \sum_{m\geq 0, p\geq 0} \varepsilon_{m,p} x_1^{p+2}x_2^{m+1}x_3^{2m+p+5}, \\
f(x_2) &= b_2 + \theta x_2x_3^2 + \sum_{p\geq 0} \theta_p x_1^{p+1}x_2x_3^{p+3} + \sum_{m\geq 0} \widetilde{\theta_m} x_2^{m+2}x_3^{2m+4} \\
&\quad + \sum_{m\geq 0, p\geq 0} \theta_{m,p} x_1^{p+1}x_2^{m+2}x_3^{2m+p+5},
\end{aligned}$$

$$\begin{aligned} f(x_3) &= \eta x_3^3 + \sum_{p\geq 0} \eta_p x_1^{p+1} x_3^{p+4} + \sum_{m\geq 0} \widetilde{\eta_m} x_2^{m+1} x_3^{2m+5} \\ &\quad + \sum_{m\geq 0, p\geq 0} \eta_{m,p} x_1^{p+1} x_2^{m+1} x_3^{2m+p+6}, \end{aligned}$$

where $b_2 \in \mathbb{C}\backslash\{0\}$ *and the other coefficients are in* $\mathbb{C}$, *and* $\frac{\alpha_p}{\gamma_p} = -\frac{q(p+1)_q}{(p+3)_q}$, $\frac{\widetilde{\alpha_m}}{\beta_m} = \frac{(3m+2)_q}{(2m+2)_q}$, $\frac{\widehat{\alpha_m}}{\widehat{\beta_m}} = \frac{(3m+3)_q}{q(2m+2)_q}$, $\frac{\alpha_{m,p}}{\beta_{m,p}} = \frac{(3m+p+4)_q}{q^{p+2}(2m+2)_q}$, $\frac{\widetilde{\alpha_m}}{\widetilde{\gamma_m}} = -\frac{q(3m+2)_q}{(m+2)_q}$, $\frac{\widehat{\alpha_m}}{\widehat{\gamma_m}} = -\frac{q(3m+3)_q}{(m+3)_q}$, $\frac{\alpha_{m,p}}{\gamma_{m,p}} = -\frac{q(3m+p+4)_q}{(p+m+4)_q}$, $\frac{\varepsilon}{\theta} = \frac{q(3)_q}{(4)_q}$, $\frac{\varepsilon_p}{\theta_p} = \frac{(p+4)_q}{q^p(4)_q}$, $\frac{\widetilde{\varepsilon_m}}{\widetilde{\theta_m}} = \frac{q(3m+6)_q}{(2m+6)_q}$, $\frac{\varepsilon_{m,p}}{\theta_{m,p}} = \frac{(3m+p+7)_q}{q^p(2m+6)_q}$, $\frac{\varepsilon}{\eta} = -q^{-1}(3)_q$, $\frac{\varepsilon_p}{\eta_p} = -\frac{(p+4)_q}{q(p+2)_q}$, $\frac{\widetilde{\varepsilon_m}}{\widetilde{\eta_p}} = -\frac{(3m+6)_q}{q(m+2)_q}$, $\frac{\varepsilon_{m,p}}{\eta_{m,p}} = -\frac{(3m+p+7)_q}{q(p+m+3)_q}$.

There are the following $U_q(sl(2))$*-module algebra structures on* $A_q(3)$

$$k(x_1) = qx_1, \quad k(x_2) = q^2 x_2, \quad k(x_3) = q^{-1} x_3, \tag{4.48}$$

$$e(x_1) = -qb_2^{-1} x_1 x_2 - \sum_{p=0}^{n} \frac{q(p+1)_q}{(p+3)_q} \alpha_p x_1^{p+3} x_3^p, \quad e(x_2) = -qb_2^{-1} x_2^2, \tag{4.49}$$

$$e(x_3) = c_{31} x_1 + e_0^{-1} x_2 x_3 + \sum_{p=0}^{n} \alpha_p x_1^{p+2} x_3^{p+1}, \tag{4.50}$$

$$f(x_1) = 0, \quad f(x_2) = b_2, \quad f(x_3) = 0, \tag{4.51}$$

where $n \in \mathbf{N}$, c_{31}, $\alpha_p \in \mathbb{C}$ *for all* p, $b_2 \in \mathbb{C}\backslash\{0\}$.

Proof. The proof is similar to that in Lemma 4.6. ■

Lemma 4.8. *For Case* $(*_{10})$, *to satisfy (3.18)-(3.19), the actions of* k, e, f *on* $A_q(3)$ *are*

$$\begin{aligned} k(x_1) &= qx_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-2} x_3, \\ e(x_1) &= \sum_{\substack{n\geq 0, p\geq 0 \\ 2+2p-n\geq 0 \\ n\neq p+1}} r_{n,p} x_1^{3+2p-n} x_2^n x_3^p + \sum_{p\geq 0} r_p x_1^{2+p} x_2^{p+1} x_3^p, \\ e(x_2) &= \sum_{\substack{n\geq 0, p\geq 0 \\ 2+2p-n\geq 0 \\ n\neq p+1}} s_{n,p} x_1^{2+2p-n} x_2^{n+1} x_3^p, \\ e(x_3) &= a_3 + \sum_{\substack{n\geq 0, p\geq 0 \\ 2+2p-n\geq 0 \\ n\neq p+1}} t_{n,p} x_1^{2+2p-n} x_2^n x_3^{p+1} + \sum_{p\geq 0} t_p x_1^{p+1} x_2^{p+1} x_3^{p+1}, \\ f(x_1) &= \sum_{\substack{n\geq 0, p\geq 0 \\ 2p-n\geq 0 \\ p\neq n+2}} u_{n,p} x_1^{2p-n+1} x_2^n x_3^{p+1} + \sum_{n\geq 0} u_n x_1^{n+5} x_2^n x_3^{n+3}, \\ f(x_2) &= \sum_{\substack{n\geq 0, p\geq 0 \\ 2p-n\geq 0 \\ p\neq n+2}} v_{n,p} x_1^{2p-n} x_2^{n+1} x_3^{p+1}, \end{aligned}$$

$$f(x_3) = \sum_{\substack{n\geq 0, p\geq 0\\ 2p-n\geq 0\\ p\neq n+2}} w_{n,p} x_1^{2p-n} x_2^n x_3^{p+2} + \sum_{n\geq 0} w_n x_1^{n+4} x_2^n x_3^{n+4},$$

where $a_3 \in \mathbb{C}\backslash\{0\}$ *and the other coefficients are in* $\mathbb{C}$, *and* $\frac{r_{n,p}}{s_{n,p}} = \frac{(n+p+1)_q}{q^{p+1}(p+1-n)_q}$, $\frac{r_{n,p}}{t_{n,p}} = -\frac{q^2(n+p+1)_q}{(2p+4)_q}$, $\frac{r_p}{t_p} = -\frac{q^2(2p+2)_q}{(2p+4)_q}$, $\frac{u_{n,p}}{v_{n,p}} = -\frac{(n+p+2)_q}{q^{p+2}(p-2-n)_q}$, $\frac{u_{n,p}}{w_{n,p}} = -\frac{(n+p+2)_q}{q(2p+2)_q}$, $\frac{u_n}{w_n} = -\frac{(2n+4)_q}{q(2n+6)_q}$.

Specifically, there are the following $U_q(sl(2))$*-module algebra structures on* $A_q(3)$

$$k(x_1) = qx_1, \quad k(x_2) = qx_2, \quad k(x_3) = q^{-2}x_3, \tag{4.52}$$

$$e(x_1) = 0, \quad e(x_2) = 0, \quad e(x_3) = a_3, \tag{4.53}$$

$$f(x_1) = a_3^{-1}x_1x_3, \quad f(x_2) = a_3^{-1}x_2x_3, \quad f(x_3) = -qa_3^{-1}x_3^2, \tag{4.54}$$

where $a_3 \in \mathbb{C}\backslash\{0\}$.

Proof. In this case, we have $\alpha_1 = q$, $\alpha_2 = q$, $\alpha_3 = q^{-2}$. Therefore, $wt(M_{ef}) \bowtie \begin{bmatrix} q^3 & q^3 & 1 \\ q^{-1} & q^{-1} & q^{-4} \end{bmatrix}$. Then, the proof is similar to those in the above lemmas. ■

Lemma 4.9. *For Case* $(*_{11})$, *to satisfy (3.18)-(3.19), the actions of* k, e *and* f *on* $A_q(3)$ *are*

$$\begin{aligned}
k(x_1) &= q^2x_1, \quad k(x_2) = q^{-1}x_2, \quad k(x_3) = q^{-1}x_3,\\
e(x_1) &= \sum_{\substack{m\geq 0, p\geq 0\\ 2m-p\geq 0\\ m\neq p+2}} \widetilde{r_{m,p}} x_1^{m+2} x_2^p x_3^{2m-p} + \sum_{p\geq 0} \widetilde{r_p} x_1^{p+4} x_2^p x_3^{p+4},\\
e(x_2) &= \sum_{\substack{m\geq 0, p\geq 0\\ 2m-p\geq 0\\ m\neq p+2}} \widetilde{s_{m,p}} x_1^{m+1} x_2^{p+1} x_3^{2m-p},\\
e(x_3) &= \sum_{\substack{m\geq 0, p\geq 0\\ 2m-p\geq 0\\ m\neq p+2}} \widetilde{t_{n,p}} x_1^{m+1} x_2^p x_3^{2m-p+1} + \sum_{p\geq 0} \widetilde{t_p} x_1^{p+3} x_2^p x_3^{p+5},\\
f(x_1) &= b_1 + \sum_{\substack{m\geq 0, p\geq 0\\ 2m+2-p\geq 0\\ p\neq m+1}} \widetilde{u_{m,p}} x_1^{m+1} x_2^p x_3^{2m+2-p} + \sum_{m\geq 0} \widetilde{u_m} x_1^{m+1} x_2^{m+1} x_3^{m+1},\\
f(x_2) &= \sum_{\substack{m\geq 0, p\geq 0\\ 2m+2-p\geq 0\\ p\neq m+1}} \widetilde{v_{m,p}} x_1^m x_2^{p+1} x_3^{2m-p+2},\\
f(x_3) &= \sum_{\substack{n\geq 0, p\geq 0\\ 2m+2-p\geq 0\\ p\neq m+1}} \widetilde{w_{m,p}} x_1^m x_2^p x_3^{2m-p+3} + \sum_{m\geq 0} \widetilde{w_m} x_1^m x_2^{m+1} x_3^{m+2},
\end{aligned}$$

where $b_1 \in \mathbb{C}\backslash\{0\}$ *and the other coefficients are in* $\mathbb{C}$, *and* $\frac{\widetilde{r_{m,p}}}{\widetilde{s_{m,p}}} = \frac{(2m+2)_q}{q^{m+1}(m-2-p)_q}$,

$\frac{\widetilde{r_{m,p}}}{\widetilde{t_{m,p}}} = -\frac{q(2m+2)_q}{(m+p+2)_q}$, $\frac{r_p}{t_p} = -\frac{q(2p+6)_q}{(2p+4)_q}$, $\frac{\widetilde{u_{m,p}}}{\widetilde{v_{m,p}}} = \frac{(2m+4)_q}{q^{m+3}(m+1-p)_q}$, $\frac{\widetilde{u_{m,p}}}{\widetilde{w_{m,p}}} = -\frac{(2m+4)_q}{q^2(m+p+1)_q}$, $\frac{\widetilde{u_m}}{\widetilde{w_m}} = -\frac{(2m+4)_q}{q^2(2m+2)_q}$.

There are the following $U_q(sl(2))$-module algebra structures on $A_q(3)$

$$k(x_1) = q^2 x_1, \quad k(x_2) = q^{-1}x_2, \quad k(x_3) = q^{-1}x_3, \tag{4.55}$$
$$e(x_1) = -qb_1^{-1}x_1^2, \quad e(x_2) = b_1^{-1}x_1x_2, \quad e(x_3) = b_1^{-1}x_1x_3, \tag{4.56}$$
$$f(x_1) = b_1, \quad f(x_2) = 0, \quad f(x_3) = 0, \tag{4.57}$$

where $b_1 \in \mathbb{C}\backslash\{0\}$.

Proof. The proof is similar to that in Lemma 4.8. ■

Next, we begin to classify all module-algebra structures of $U_q(sl(3)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{i=1,2}$ on $A_q(3)$ when $k_i \in \mathrm{Aut\ L}(A_q(3))$ for $i = 1, 2$.

For $U_q(sl(3))$, there are two sub-Hopf algebras which are isomorphic to $U_q(sl(2))$. One is generated by k_1, e_1 and f_1. Denote this algebra by A. The other one, denoted by B, is generated by k_2, e_2 and f_2. By the definition of module algebra of one Hopf algebra, the module-algebra structures on $A_q(2)$ of these two sub-Hopf algebras are of the kinds discussed above.

Denote 9 cases of the actions of k_1, e_1, f_1 (resp. k_2, e_2, f_2) in Lemma 4.1-Lemma 4.9 by $(A1)$, $\cdots$, $(A9)$ (resp. $(B1)$, $\cdots$, $(B9)$). To determine all module-algebra structures of $U_q(sl(3)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{i=1,2}$ on $A_q(3)$ when $k_i \in \mathrm{Aut\ L}(A_q(3))$ for $i = 1, 2$, we have to find all the actions of k_1, e_1, f_1 and k_2, e_2, f_2 which are compatible, i.e., the following equalities hold

$$k_1e_2(u) = q^{-1}e_2k_1(u), \quad k_1f_2(u) = qf_2k_1(u), \tag{4.58}$$
$$k_2e_1(u) = q^{-1}e_1k_2(u), \quad k_2f_1(u) = qf_1k_2(u), \tag{4.59}$$
$$e_1f_2(u) = f_2e_1(u), \quad e_2f_1(u) = f_1e_2(u), \tag{4.60}$$
$$e_1^2e_2(u) - (q+q^{-1})e_1e_2e_1(u) + e_2e_1^2(u) = 0, \tag{4.61}$$
$$e_2^2e_1(u) - (q+q^{-1})e_2e_1e_2(u) + e_1e_2^2(u) = 0, \tag{4.62}$$
$$f_1^2f_2(u) - (q+q^{-1})f_1f_2f_1(u) + f_2f_1^2(u) = 0, \tag{4.63}$$
$$f_2^2f_1(u) - (q+q^{-1})f_2f_1f_2(u) + f_1f_2^2(u) = 0, \tag{4.64}$$

and $e_if_i(u) - f_ie_i(u) = \frac{k_i - k_i^{-1}}{q-q^{-1}}(u)$ holds for $u \in \{x_1, x_2, x_3\}$ and $i \in \{1, 2\}$.

Because the actions of k_1, e_1, f_1 and k_2, e_2, f_2 in $U_q(sl(3))$ are symmetric, we only need to check 45 cases, i.e., whether (Ai) is compatible with (Bj) for any $1 \le i \le j \le 9$. We use $(Ai)|(B_j)$ to denote that the actions of k_1, e_1, f_1 are those in (Ai) and the actions of k_2, e_2, f_2 are those in (Bj). Moreover, in Case $(Aj)|(Bj)$ $(j \ge 2)$, since the actions of e_i, f_i are not zero simultaneously for $i \in \{1, 2\}$, (4.58) and (4.59) can not be satisfied simultaneously. Therefore, the Cases $(Aj)|(Bj)$ $(j \ge 2)$ should be excluded.

First, let us consider Case $(A2)|(B5)$. Since $k_2e_1(x_1) = k_2(a_1) = a_1$, $q^{-1}e_1k_2(x_1) = q^{-1}a_1$ and $a_1 \ne 0$, $k_2e_1(x_1) = q^{-1}e_1k_2(x_1)$ does not hold. Therefore, $(A2)|(B5)$ should be excluded. For the same reason, we exclude $(A2)|(B6)$, $(A2)|(B8)$, $(A2)|(B9)$, $(A3)|(B4)$, $(A3)|(B7)$, $(A3)|(B8)$, $(A3)|(B9)$, $(A4)|(B6)$,

$(A4)|(B7)$, $(A4)|(B8)$, $(A4)|(B9)$, $(A5)|(B7)$, $(A5)|(B8)$, $(A5)|(B9)$, $(A6)|(B7)$, $(A6)|(B8)$, $(A6)|(B9)$, $(A7)|(B8)$, $(A7)|(B9)$, $(A8)|(B9)$.

Second, we consider $(A1)|(B2)$. Since $k_1f_2(x_1) = -qa_1^{-1}x_1^2$ and $qf_2k_1(x_1) = \mp q^2a_1^{-1}x_1^2$, we have $k_1f_2(x_1) \neq qf_2k_1(x_1)$. Thus, $(A_1)|(B2)$ should be excluded. Similarly, $(A1)|(Bi)$ should be excluded for $i \geq 3$.

Therefore, we only need to consider the following cases: $(A1)|(B1)$, $(A2)|(B3)$, $(A2)|(B4)$, $(A2)|(B7)$, $(A3)|(B5)$, $(A3)|(B6)$, $(A4)|(B5)$, $(A4)|(B7)$, $(A5)|(B6)$.

Lemma 4.10. *For Case $(A1)|(B1)$, all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows*

$$k_i(x_j) = \pm x_j, e_i(x_j) = 0, f_i(x_j) = 0,$$

for $i \in \{1,2\}$, $j \in \{1,2,3\}$, which are pairwise non-isomorphic.

Proof. It can be seen that (4.58)-(4.64) are satisfied for any $u \in \{x_1, x_2, x_3\}$ in this case. Therefore, they are module-algebra structures of $U_q(sl(3))$ on $A_q(3)$. Since all the automorphisms of $A_q(3)$ commute with the actions of k_1 and k_2, all these module-algebra structures are pairwise non-isomorphic. ∎

Lemma 4.11. *For Case $(A2)|(B3)$, all $U_q(sl(3))$-module algebra structures on $A_q(3)$ are as follows:*

$$\begin{aligned}
&k_1(x_1) = q^{-2}x_1, \quad k_1(x_2) = q^{-1}x_2, \quad k_1(x_3) = q^{-1}x_3,\\
&k_2(x_1) = qx_1, \quad k_2(x_2) = qx_2, \quad k_2(x_3) = q^2x_3,\\
&e_1(x_1) = a_1, \quad e_1(x_2) = 0, \quad e_1(x_3) = 0,\\
&e_2(x_1) = -qb_3^{-1}x_1x_3, \quad e_2(x_2) = -qb_3^{-1}x_2x_3, \quad e_2(x_3) = -qb_3^{-1}x_3^2,\\
&f_1(x_1) = -qa_1^{-1}x_1^2, \quad f_1(x_2) = -qa_1^{-1}x_1x_2, \quad f_1(x_3) = -qa_1^{-1}x_1x_3,\\
&f_2(x_1) = 0, \quad f_2(x_2) = 0, \quad f_2(x_3) = b_3,
\end{aligned}$$

where a_1, $b_3 \in \mathbb{C}\backslash\{0\}$.

All these structures are isomorphic to that with $a_1 = b_3 = 1$.

Proof. By Lemma 4.2 and Lemma 4.3, to determine the module-algebra structures of $U_q(sl(3))$ on $A_q(3)$, we have to make (4.58)-(4.64) hold for any $u \in \{x_1, x_2, x_3\}$ using the actions of k_1, e_1, f_1 in Lemma 4.2 and the actions of k_2, e_2, f_2 in Lemma 4.3.

Since $k_1e_2(x_1) = q^{-1}e_2k_1(x_1) = q^{-3}e_2(x_1)$, we have $e_2(x_1) = -qb_3^{-1}x_1x_3$, i.e., $\mu_1 = \mu_2 = \mu_3 = 0$. Using $k_1e_2(x_2) = q^{-1}e_2k_1(x_2) = q^{-2}e_2(x_2)$, we obtain $e_2(x_2) = -qb_3^{-1}x_2x_3$. Similarly, by $k_2f_1(x_2) = qf_1k_2(x_2) = q^2f_1(x_2)$ and $k_2f_1(x_3) = qf_1k_2(x_3) = q^3f_1(x_3)$, we get $f_1(x_2) = -qa_1^{-1}x_1x_2$ and $f_1(x_3) = -qa_1^{-1}x_1x_3$. Then, it is easy to check that (4.58)-(4.59) hold for any $u \in \{x_1, x_2, x_3\}$.

Then, we check that (4.60) holds. Obviously, $e_1f_2(u) = f_2e_1(u)$ for any

$u \in \{x_1, x_2, x_3\}$. Now, we check $e_2 f_1(x_1) - f_1 e_2(x_1) = 0$. In fact,

$$
\begin{aligned}
& e_2 f_1(x_1) - f_1 e_2(x_1) \\
= \ & e_2(-q a_1^{-1} x_1^2) - f_1(-q b_3^{-1} x_1 x_3) \\
= \ & -q a_1^{-1}(x_1 e_2(x_1) + e_2(x_1) k_2(x_1)) + q b_3^{-1}(k_1^{-1}(x_1) f_1(x_3) + f_1(x_1) x_3) \\
= \ & (q^2 + q^4) a_1^{-1} b_3^{-1} x_1^2 x_3 - (q^4 + q^2) a_1^{-1} b_3^{-1} x_1^2 x_3 \\
= \ & 0.
\end{aligned}
$$

Similarly, other equalities in (4.60) can be checked.

Next, we check that (4.61)-(4.64) hold for any $u \in \{x_1, x_2, x_3\}$. We only check

$$
e_2^2 e_1(x_1) - (q + q^{-1}) e_2 e_1 e_2(x_1) + e_1 e_2^2(x_1) = 0.
$$

In fact,

$$
\begin{aligned}
& e_2^2 e_1(x_1) - (q + q^{-1}) e_2 e_1 e_2(x_1) + e_1 e_2^2(x_1) \\
= \ & 0 - (q + q^{-1}) e_2 e_1(-q b_3^{-1} x_1 x_3) + e_1 e_2(-q b_3^{-1} x_1 x_3) \\
= \ & (q + q^{-1}) b_3^{-1} e_2(a_1 x_3) - q b_3^{-1} e_1(-q b_3^{-1} x_1 x_3^2 - q^3 b_3^{-1} x_1 x_3^2) \\
= \ & -q(q + q^{-1}) a_1 b_3^{-2} x_3^2 + a_1 b_3^{-2}(1 + q^2) x_3^2 \\
= \ & 0.
\end{aligned}
$$

The other equalities can be checked similarly.

Finally, we claim that all the actions with nonzero a_1 and b_3 are isomorphic to the specific action with $a_1 = 1$, $b_3 = 1$. The desired isomorphism is given by $\psi_{a_1, b_3} : x_1 \mapsto a_1 x_1, x_2 \mapsto x_2, x_3 \mapsto b_3 x_3$. ■

Lemma 4.12. *For Case (A2)|(B4), all $U_q(sl(3))$-module algebra structures on $A_q(3)$ are as follows*

$$
\begin{aligned}
& k_1(x_1) = q^{-2} x_1, \quad k_1(x_2) = q^{-1} x_2, \quad k_1(x_3) = q^{-1} x_3, \\
& k_2(x_1) = q x_1, \quad k_2(x_2) = q^{-1} x_2, \quad k_2(x_3) = x_3, \\
& e_1(x_1) = a_1, \quad e_1(x_2) = 0, \quad e_1(x_3) = 0, \\
& e_2(x_1) = 0, \quad e_2(x_2) = c_{21} x_1, \quad e_2(x_3) = 0, \\
& f_1(x_1) = -q a_1^{-1} x_1^2, \quad f_1(x_2) = -q a_1^{-1} x_1 x_2, \quad f_1(x_3) = -q a_1^{-1} x_1 x_3, \\
& f_2(x_1) = c_{21}^{-1} x_2, \quad f_2(x_2) = 0, \quad f_2(x_3) = 0,
\end{aligned}
$$

where a_1, $c_{21} \in \mathbb{C} \backslash \{0\}$.

All these module-algebra structures are isomorphic to that with $a_1 = c_{21} = 1$.

Proof. By the above actions of k_1, e_1, f_1 and k_2, e_2, f_2, it is easy to check that (4.58)-(4.64) hold for any $u \in \{x_1, x_2, x_3\}$. Therefore, by Lemma 4.2 and Lemma 4.4, they determine the module-algebra structures of $U_q(sl(3))$ on $A_q(3)$.

Next, we prove that there are no other actions except these in this lemma.

Using $k_1 e_2(x_1) = q^{-1} e_2 k_1(x_1) = q^{-3} e_2(x_1)$, we can obtain $e_2(x_1) = 0$. By Lemma 4.4, we also have $e_2(x_2) = c_{21} x_1$ and $e_2(x_3) = 0$. Similarly, by $k_1 f_2(x_1) = q f_2 k_1(x_1) = q^{-1} f_2(x_1)$, we get $f_2(x_1) = d_{12} x_2$. Therefore, $f_2(x_2) = f_2(x_3) = 0$.

Then, using $e_2f_2(x_i) - f_2e_2(x_i) = \frac{k_2-k_2^{-1}}{q-q^{-1}}(x_i)$ for any $i \in \{1,2,3\}$, we obtain $d_{12} = c_{21}^{-1}$. Since $k_2f_1(x_2) = qf_1k_2(x_2) = f_1(x_2)$ and $k_2f_1(x_3) = qf_1k_2(x_3) = qf_1(x_3)$, by Lemma 4.2, we have $f_1(x_2) = -qa_1^{-1}x_1x_2 + \xi_3x_3^3$, $f_1(x_3) = -qa_1^{-1}x_1x_3$.

Due to the conditions of the module algebra, it is easy to see that we have to let $f_1^2f_2(x_3) - (q+q^{-1})f_1f_2f_1(x_3) + f_2f_1^2(x_3) = 0$ hold. On the other hand, we have

$$\begin{aligned}
& f_1^2f_2(x_3) - (q+q^{-1})f_1f_2f_1(x_3) + f_2f_1^2(x_3) \\
= \;& -(q+q^{-1})f_1f_2(-qa_1^{-1}x_1x_3) + f_2f_1(-qa_1^{-1}x_1x_3) \\
= \;& q(q+q^{-1})a_1^{-1}c_{21}^{-1}f_1(x_2x_3) - qa_1^{-1}f_2(f_1(x_1)x_3 + q^2x_1f_1(x_3)) \\
= \;& q(q+q^{-1})a_1^{-1}c_{21}^{-1}(f_1(x_2)x_3 + qx_2f_1(x_3)) + qa_1^{-2}(q+q^3)f_2(x_1^2x_3) \\
= \;& (q^2+1)a_1^{-1}c_{21}^{-1}(-(q+q^3)a_1^{-1}x_1x_2x_3 + \xi_3x_3^4) \\
& +(q^2+1)(q+q^3)c_{21}^{-1}a_1^{-2}x_1x_2x_3 \\
= \;& (q^2+1)a_1^{-1}c_{21}^{-1}\xi_3x_3^4.
\end{aligned}$$

Hence, we get $\xi_3 = 0$. Therefore, $f_1(x_2) = -qa_1^{-1}x_1x_2$. Thus, there are no other actions except those in this lemma.

Finally, we claim that all the actions with nonzero a_1 and c_{21} are isomorphic to the specific action with $a_1 = 1$, $c_{21} = 1$. The desired isomorphism is given by $\psi_{a_1,c_{21}} : x_1 \mapsto a_1x_1, x_2 \mapsto a_1c_{21}x_2, x_3 \mapsto x_3$. ■

Lemma 4.13. *For Case $(A2)|(B7)$, all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows*

$$\begin{aligned}
& k_1(x_1) = q^{-2}x_1, \quad k_1(x_2) = q^{-1}x_2, \quad k_1(x_3) = q^{-1}x_3, \\
& k_2(x_1) = qx_1, \quad k_2(x_2) = q^2x_2, \quad k_2(x_3) = q^{-1}x_3, \\
& e_1(x_1) = a_1, \quad e_1(x_2) = 0, \quad e_1(x_3) = 0, \\
& e_2(x_1) = -qb_2^{-1}x_1x_2, \quad e_2(x_2) = -qb_2^{-1}x_2^2, \quad e_2(x_3) = c_{31}x_1 + b_2^{-1}x_2x_3, \\
& f_1(x_1) = -qa_1^{-1}x_1^2, \quad f_1(x_2) = -qa_1^{-1}x_1x_2, \quad f_1(x_3) = -qa_1^{-1}x_1x_3 + \xi_4x_2x_3^2, \\
& f_2(x_1) = 0, \quad f_2(x_2) = b_2, \quad f_2(x_3) = 0,
\end{aligned}$$

where a_1, b_2, c_{31}, $\xi_4 \in \mathbb{C}\backslash\{0\}$ and $c_{31}\xi_4 = -qb_2^{-1}a_1^{-1}$.

All module-algebra structures above are isomorphic to that with $a_1 = b_2 = c_{31} = 1$ and $\xi_4 = -q$.

Proof. By the above actions of k_1, e_1, f_1 and k_2, e_2, f_2, it is easy to check that (4.58)-(4.64) hold for any $u \in \{x_1, x_2, x_3\}$. Therefore, by Lemma 4.2 and Lemma 4.7, they determine the module-algebra structures of $U_q(sl(3))$ on $A_q(3)$.

Next, we prove that there are no other actions except those in this lemma.

By (4.58), we can immediately obtain that $e_2(x_1) = \widetilde{\alpha_0}x_1x_2$, $e_2(x_2) = \widetilde{\beta_0}x_2^2$, $e_2(x_3) = c_{31}x_1 + \widetilde{\gamma_0}x_2x_3$, $f_2(x_1) = 0$, $f_2(x_2) = b_2$ and $f_2(x_3) = 0$. By Lemma 4.7, $\widetilde{\alpha_0} = \widetilde{\beta_0} = -q\widetilde{\gamma_0}$. According to $e_2f_2(x_1) - f_2e_2(x_1) = \frac{k_2-k_2^{-1}}{q-q^{-1}}(x_1)$, we obtain $\widetilde{\gamma_0} = b_2^{-1}$. Similarly, by (4.59), we have $f_1(x_1) = -qa_1^{-1}x_1^2$, $f_1(x_2) = -qa_1^{-1}x_1x_2$ and $f_1(x_3) = -qa_1^{-1}x_1x_3 + \xi_4x_2x_3^2$.

Next, let us consider the condition $e_2f_1(x_3)-f_1e_2(x_3)=0$. Since

$$\begin{aligned}
& e_2f_1(x_3)-f_1e_2(x_3)\\
= \;& e_2(-qa_1^{-1}x_1x_3+\xi_4x_2x_3^2)-f_1(c_{31}x_1+b_2^{-1}x_2x_3)\\
= \;& -qa_1^{-1}(x_1e_2(x_3)+e_2(x_1)k_2(x_3))+\xi_4(x_2x_3e_2(x_3)+x_2e_2(x_3)k_2(x_3)\\
& +e_2(x_2)k_2(x_3)k_2(x_3))+qc_{31}a_1^{-1}x_1^2-b_2^{-1}(k_1^{-1}(x_2)f_1(x_3)+f_1(x_2)x_3)\\
= \;& \xi_4c_{31}(q^2+1)x_1x_2x_3+a_1^{-1}b_2^{-1}(q^3+q)x_1x_2x_3\\
= \;& 0,
\end{aligned}$$

we obtain $\xi_4c_{31}=-qa_1^{-1}b_2^{-1}$.

Therefore, there are no other actions except those in this lemma.

Finally, we show that all the actions with nonzero a_1, c_{31}, b_2 and ξ_4 are isomorphic to the specific action with $a_1=b_2=c_{31}=1$ and $\xi_4=-q$. The desired isomorphism is given by $\psi_{a_1,c_{31},b_2}: x_1\mapsto a_1x_1, x_2\mapsto b_2x_2, x_3\mapsto a_1c_{31}x_3$. ■

Lemma 4.14. *For Case* $(A3)|(B5)$, *all module-algebra structures of* $U_q(sl(3))$ *on* $A_q(3)$ *are as follows:*

$$\begin{aligned}
& k_1(x_1)=qx_1,\quad k_1(x_2)=qx_2,\quad k_1(x_3)=q^2x_3,\\
& k_2(x_1)=x_1,\quad k_2(x_2)=qx_2,\quad k_2(x_3)=q^{-1}x_3,\\
& e_1(x_1)=-qb_3^{-1}x_1x_3,\quad e_1(x_2)=-qb_3^{-1}x_2x_3,\quad e_1(x_3)=-qb_3^{-1}x_3^2,\\
& e_2(x_1)=0,\quad e_2(x_2)=0,\quad e_2(x_3)=c_{32}x_2,\\
& f_1(x_1)=0,\quad f_1(x_2)=0,\quad f_1(x_3)=b_3,\\
& f_2(x_1)=0,\quad f_2(x_2)=c_{32}^{-1}x_3,\quad f_2(x_3)=0,
\end{aligned}$$

where b_3, $c_{32}\in\mathbb{C}\backslash\{0\}$.

All module-algebra structures above are isomorphic to that with $b_3=c_{32}=1$.

Proof. The proof is similar to that in Lemma 4.12. ■

Lemma 4.15. *For Case* $(A3)|(B6)$, *all module-algebra structures of* $U_q(sl(3))$ *on* $A_q(3)$ *are as follows:*

$$\begin{aligned}
& k_1(x_1)=qx_1,\quad k_1(x_2)=qx_2,\quad k_1(x_3)=q^2x_3,\\
& k_2(x_1)=qx_1,\quad k_2(x_2)=q^{-2}x_2,\quad k_2(x_3)=q^{-1}x_3,\\
& e_1(x_1)=-qb_3^{-1}x_1x_3+\mu_1x_1^2x_2,\quad e_1(x_2)=-qb_3^{-1}x_2x_3,\quad e_1(x_3)=-qb_3^{-1}x_3^2,\\
& e_2(x_1)=0,\quad e_2(x_2)=a_2,\quad e_2(x_3)=0,\\
& f_1(x_1)=0,\quad f_1(x_2)=0,\quad f_1(x_3)=b_3,\\
& f_2(x_1)=d_{13}x_3+a_2^{-1}x_1x_2,\quad f_2(x_2)=-qa_2^{-1}x_2^2,\quad f_2(x_3)=-qa_2^{-1}x_2x_3,
\end{aligned}$$

where d_{13}, a_2, μ_1, $b_3\in\mathbb{C}\backslash\{0\}$ *and* $\mu_1d_{13}=-qa_2^{-1}b_3^{-1}$.

All module-algebra structures above are isomorphic to that with $d_{13}=a_2=b_3=1$ *and* $\mu_1=-q$.

Proof. The proof is similar to that in Lemma 4.13. ∎

Lemma 4.16. *For Case $(A4)|(B5)$, all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows:*

$$\begin{aligned}
&k_1(x_1) = qx_1, \quad k_1(x_2) = q^{-1}x_2, \quad k_1(x_3) = x_3,\\
&k_2(x_1) = x_1, \quad k_2(x_2) = qx_2, \quad k_2(x_3) = q^{-1}x_3,\\
&e_1(x_1) = 0, \quad e_1(x_2) = c_{21}x_1, \quad e_1(x_3) = 0,\\
&e_2(x_1) = 0, \quad e_2(x_2) = 0, \quad e_2(x_3) = c_{32}x_2,\\
&f_1(x_1) = c_{21}^{-1}x_2, \quad f_1(x_2) = 0, \quad f_1(x_3) = 0,\\
&f_2(x_1) = 0, \quad f_2(x_2) = c_{32}^{-1}x_3, \quad f_2(x_3) = 0,
\end{aligned}$$

where c_{21}, $c_{32} \in \mathbb{C}\backslash\{0\}$.

All the above module-algebra structures are isomorphic to that with $c_{21} = c_{32} = 1$.

Proof. By the actions of k_1, e_1, f_1 and k_2, e_2, f_2, it is easy to check that (4.58)-(4.64) hold for any $u \in \{x_1, x_2, x_3\}$. Therefore, by Lemma 4.4 and Lemma 4.5, they determine the module-algebra structures of $U_q(sl(3))$ on $A_q(3)$.

Next, we prove that there are no other actions except these in this lemma.

By Lemma 4.5 and using that (4.58) holds for any $u \in \{x_1, x_2, x_3\}$, we can obtain that $e_2(x_1) = \sum_{n\geq 0} \widetilde{a_n} x_1^{n+2}x_2^{n+2}x_3^n$, $e_2(x_2) = 0$, $e_2(x_3) = c_{32}x_2 + \sum_{n\geq 0} \widetilde{c_n} x_1^{n+1}x_2^{n+2} \cdot x_3^{n+1}$, $f_2(x_1) = \sum_{m\geq 0} \widetilde{d_{m,m+1}} x_1^{m+2}x_2^m x_3^{m+2}$, $f_2(x_2) = d_{23}x_3 + \sum_{m\geq 0} \widetilde{e_{m,m+1}} x_1^{m+1}x_2^{m+1}x_3^{m+2}$, $f_2(x_3) = \sum_{m\geq 0} \widetilde{g_{m,m+1}} x_1^{m+1}x_2^m x_3^{m+3}$.

By Lemma 4.5, we know that $\frac{\widetilde{a_n}}{\widetilde{c_n}} = -\frac{q(2n+2)_q}{(2n+4)_q}$, $\frac{\widetilde{d_{m,m+1}}}{\widetilde{e_{m,m+1}}} = \frac{(2m+2)_q}{q^m(2)_q}$, $\frac{\widetilde{d_{m,m+1}}}{\widetilde{g_{m,m+1}}} = \frac{q^{2m+2}-1}{1-q^{2m+2}} = -1$. Set $v_n = \frac{\widetilde{a_n}}{\widetilde{c_n}}$ and $\kappa_m = \frac{\widetilde{d_{m,m+1}}}{\widetilde{e_{m,m+1}}}$.

Next, we consider $e_2f_2(x_2) - f_2e_2(x_2) = \frac{k_2-k_2^{-1}}{q-q^{-1}}(x_2) = x_2$. By some computations, we obtain

$$\begin{aligned}
&e_2f_2(x_2) - f_2e_2(x_2)\\
&= c_{32}d_{23}x_2 + \sum_{n\geq 0}(q^{2n+4}-1)d_{23}v_n x_1^{n+1}x_2^{n+2}x_3^{n+1}\\
&-\sum_{m\geq 0}(q^{-1}-q^{2m+3})c_{32}\kappa_m x_1^{m+1}x_2^{m+2}x_3^{m+1} + \sum_{m\geq 0, n\geq 0} q^{3mn+3m+3n+2}(1-q^2)\\
&\cdot(1-q^{2m+2n+6})v_n\kappa_m x_1^{m+n+2}x_2^{m+n+3}x_3^{m+n+2}.
\end{aligned}$$

If there exist v_n and κ_m not equal to zero, we can choose the terms with coefficients v_{n_e} and κ_{m_f} in $e_2(x_1)$, $e_2(x_2)$, $e_2(x_3)$, $f_2(x_1)$, $f_2(x_2)$, $f_2(x_3)$ such that their degrees are highest. Then, the unique monomial of the highest degree in $(e_2f_2 - f_2e_2)(x_2)$ is

$$q^{3m_fn_e+3m_f+3n_e+2}(1-q^2)(1-q^{2m_f+2n_e+6})v_{n_e}\kappa_{m_f}x_1^{m_f+n_e+2}x_2^{m_f+n_e+3}x_3^{m_f+n_e+2}.$$

Since the degree of this term is larger than 1, this case is impossible. Similarly, all cases except that all v_n, κ_m are equal to zero should be excluded. Therefore, we obtain that $e_2(x_1)=0$, $e_2(x_2)=0$, $e_2(x_3)=c_{32}x_2$, $f_2(x_1)=0$, $f_2(x_2)=c_{32}^{-1}x_3$ and $f_2(x_3)=0$.

Similarly, using (4.59), Lemma 4.4 and $e_1f_1(u)-f_1e_1(u)=\frac{k_1-k_1^{-1}}{q-q^{-1}}(u)$ for any $u\in\{x_1,x_2,x_3\}$, we can obtain $e_1(x_1)=0$, $e_1(x_2)=c_{21}x_1$, $e_1(x_3)=0$, $f_1(x_1)=c_{21}^{-1}x_2$, $f_1(x_2)=0$, $f_1(x_3)=0$.

Therefore, there are no other actions except the ones in this lemma.

Finally, we claim that all the actions with nonzero c_{21}, c_{32} are isomorphic to the specific action with $c_{21}=c_{32}=1$. The desired isomorphism is given by $\psi_{c_{21},c_{32}}:x_1\mapsto x_1, x_2\mapsto c_{21}x_2, x_3\mapsto c_{21}c_{32}x_3$. ■

Lemma 4.17. *For Case* $(A5)|(B6)$, *all module-algebra structures of* $U_q(sl(3))$ *on* $A_q(3)$ *are*

$$\begin{aligned}
&k_1(x_1)=x_1,\quad k_1(x_2)=qx_2,\quad k_1(x_3)=q^{-1}x_3,\\
&k_2(x_1)=qx_1,\quad k_2(x_2)=q^{-2}x_2,\quad k_2(x_3)=q^{-1}x_3,\\
&e_1(x_1)=0,\quad e_1(x_2)=0,\quad e_1(x_3)=c_{32}x_2,\\
&e_2(x_1)=0,\quad e_2(x_2)=a_2,\quad e_2(x_3)=0,\\
&f_1(x_1)=0,\quad f_1(x_2)=c_{32}^{-1}x_3,\quad f_1(x_3)=0,\\
&f_2(x_1)=a_2^{-1}x_1x_2,\quad f_2(x_2)=-qa_2^{-1}x_2^2,\quad f_2(x_3)=-qa_2^{-1}x_2x_3,
\end{aligned}$$

where c_{32}, $a_2\in\mathbb{C}\backslash\{0\}$.

All module-algebra structures are isomorphic to that with $a_2=c_{32}=1$.

Proof. It is easy to check that the above actions of k_1, e_1, f_1 and k_2, e_2, f_2 determine module-algebra structures of $U_q(sl(3))$ on $A_q(3)$.

Then we will prove that there are no other actions except for those in this lemma.

By (4.58) for any $u\in\{x_1,x_2,x_3\}$ and Lemma 4.6, we have

$$\begin{aligned}
e_2(x_1)&=(q-q^3)ux_1^4x_3+\sum_{n\geq 0}(q-q^{2n+5})v_nx_1^{3n+7}x_2^{n+1}x_3^{n+2},\\
e_2(x_2)&=a_2+(q^4-1)ux_1^3x_2x_3+\sum_{n\geq 0}(q^{3n+7}-q^{n+1})v_nx_1^{3n+6}x_2^{n+2}x_3^{n+2},\\
e_2(x_3)&=(q^4-1)ux_1^3x_3^2+\sum_{n\geq 0}(q^{4n+8}-1)v_nx_1^{3n+6}x_2^{n+1}x_3^{n+3},\\
f_2(x_1)&=gx_1x_2+(q^3-q^{-1})\varepsilon x_1^4x_2^2x_3+\sum_{p\geq 0}(q^{2p+5}-q^{-1})\mu_px_1^{3p+7}x_2^{p+3}x_3^{p+2},\\
f_2(x_2)&=-qgx_2^2+(q-q^5)\varepsilon x_1^3x_2^3x_3+\sum_{p\geq 0}(q^{p+2}-q^{3p+8})\mu_px_1^{3p+6}x_2^{p+4}x_3^{p+2},\\
f_2(x_3)&=-qgx_2x_3+(1-q^6)\varepsilon x_1^3x_2^2x_3^2+\sum_{p\geq 0}(1-q^{4p+10})\mu_px_1^{3p+6}x_2^{p+3}x_3^{p+3}.
\end{aligned}$$

Then we will consider the condition $e_2f_2(u)-f_2e_2(u)=\frac{k_2-k_2^{-1}}{q-q^{-1}}(u)$ for any $u\in\{x_1,x_2,x_3\}$.

Let us assume that there exist some u or v_n which are not equal to zero. Then, we can choose the monomials in $e_2(x_1)$, $e_2(x_2)$, $e_2(x_3)$ with the highest degree. Obviously, these monomials are unique. It is also easy to see that $f(x_1)$, $f(x_2)$, $f(x_3)$ can not be equal to zero simultaneously. Therefore, there are some nonzero g, ε or μ_p. Similarly, those monomials in $f_2(x_1)$, $f_2(x_2)$, $f_2(x_3)$ with the highest degree are chosen. Then, by some computations, we can obtain a monomial with the highest degree, whose degree is larger than 1. Then, we get a contradiction with $e_2f_2(x_1)-f_2e_2(x_1)=x_1$. For example, if the coefficient of the monomials in $e_2(x_1)$, $e_2(x_2)$ and $e_2(x_3)$ is v_{n_e} and the coefficient of the monomials in $f_2(x_1)$, $f_2(x_2)$, $f_2(x_3)$ with the highest degree is μ_{p_f}, then the monomial with the highest degree in $e_2f_2(x_1)-f_2e_2(x_1)$ is

$$q^{7n_ep_f+15n_e+11p_f+22}(1-q^{2n_e+2p_f+10})^2 v_{n_e}\mu_{p_f}x_1^{3n_e+3p_f+13}x_2^{n_e+p_f+4}x_3^{n_e+p_f+4}.$$

Therefore, all u, v_n are equal to zero. Then, $e_2(x_1)=0$, $e_2(x_2)=a_2$ and $e_2(x_3)=0$. Thus, we can obtain

$$\begin{aligned}&e_2f_2(x_1)-f_2e_2(x_1)\\ &= ga_2x_1+(1-q^{-4})(1+q^2)\varepsilon a_2x_1^4x_2x_3+\textstyle\sum_{p\geq 0}\frac{1-q^{2p+6}}{1-q^2}(q^{-1-p}-q^{-3p-7})a_2\mu_px_1^{3p+7}x_2^{p+2}x_3^{p+2}.\end{aligned}$$

Thus, we also obtain $f_2(x_1)=a_2^{-1}x_1x_2$, $f_2(x_2)=-qa_2^{-1}x_2^2$ and $f_2(x_3)=-qa_2^{-1}x_2x_3$.

On the other hand, by a similar discussion, from (4.59), Lemma 4.5 and $e_1f_1(u)-f_1e_1(u)=\frac{k_1-k_1^{-1}}{q-q^{-1}}(u)$ for any $u\in\{x_1,x_2,x_3\}$, we can obtain $e_1(x_1)=0$, $e_1(x_2)=0$, $e_1(x_3)=c_{32}x_2$, $f_1(x_1)=0$, $f_1(x_2)=c_{32}^{-1}x_3$, $f_1(x_3)=0$.

Therefore, there are no other actions except those in this lemma.

Finally, we claim that all module algebra structures with nonzero a_2, c_{32} are isomorphic to that with $a_2=c_{32}=1$. The desired isomorphism is given by

$$\psi_{a_2,c_{32}}: x_1\mapsto x_1, x_2\mapsto a_2x_2, x_3\mapsto a_2c_{32}x_3. \qquad \blacksquare$$

Lemma 4.18. *For Case $(A4)|(B7)$, all module-algebra structures of $U_q(sl(3))$ on $A_q(3)$ are as follows*

$$\begin{aligned}
&k_1(x_1)=qx_1, \quad k_1(x_2)=q^{-1}x_2, \quad k_1(x_3)=x_3,\\
&k_2(x_1)=qx_1, \quad k_2(x_2)=q^2x_2, \quad k_2(x_3)=q^{-1}x_3,\\
&e_1(x_1)=0, \quad e_1(x_2)=c_{21}x_1, \quad e_1(x_3)=0,\\
&e_2(x_1)=-qb_2^{-1}x_1x_2, \quad e_2(x_2)=-qb_2^{-1}x_2^2, \quad e_2(x_3)=b_2^{-1}x_2x_3,\\
&f_1(x_1)=c_{21}^{-1}x_2, \quad f_1(x_2)=0, \quad f_1(x_3)=0,\\
&f_2(x_1)=0, \quad f_2(x_2)=b_2, \quad f_2(x_3)=0,
\end{aligned}$$

where b_2, $c_{21}\in\mathbb{C}\backslash\{0\}$.

All module-algebra structures are isomorphic to that with $b_2=c_{21}=1$.

Proof. The proof is similar to that in Lemma 4.17. $\blacksquare$

5. Classification of $U_q(sl(m+1))$-module algebra structures on $A_q(3)$ and $A_q(2)$

In this section, we will present the classification of $U_q(sl(m+1))$-module algebra structures on $A_q(3)$ and similarly, on $A_q(2)$ for $m\geq 2$.

The associated classical limit actions of sl_3 (which here is the Lie algebra generated by h_1, h_2, e_1, e_2, f_1, f_2 with the relations $[e_1, f_1] = h_1$, $[e_2, f_2] = h_2$, $[e_1, f_2] = [e_2, f_1] = 0$, $[h_1, e_1] = 2e_1$, $[h_1, e_2] = -e_2$, $[h_2, e_2] = 2e_2$, $[h_2, e_1] = -e_1$, $[h_1, f_1] = -2f_1$, $[h_1, f_2] = f_2$, $[h_2, f_2] = -2f_2$, $[h_2, f_1] = f_1$, $[h_1, h_2] = 0$) on $\mathbb{C}[x_1, x_2, x_3]$ by differentiations are derived from the quantum actions via substituting $k_1 = q^{h_1}$, $k_2 = q^{h_2}$ with a subsequent formal passage to the limit as $q \to 1$.

Since the actions of k_1, e_1, f_1 and k_2, e_2, f_2 in $U_q(sl(3))$ are symmetric, by Lemma 4.10-Lemma 4.18 and the discussion above, we obtain the following theorem.

Theorem 5.1. *$U_q(sl(3)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{i=1,2}$-module algebra structures up to isomorphisms on $A_q(3)$ when $k_i \in Aut\ L(A_q(3))$ for $i = 1, 2$ and their classical limits, i.e., Lie algebra sl_3-actions by differentiations on $\mathbb{C}[x_1, x_2, x_3]$ are as follows:*

$U_q(sl(3))$-module algebra structures	Classical limit sl_3-actions on $\mathbb{C}[x_1, x_2, x_3]$
$k_i(x_s) = \pm x_s$, $k_j(x_s) = \pm x_s$, $e_i(x_s) = 0$, $e_j(x_s) = 0$, $f_i(x_s) = 0$, $f_j(x_s) = 0$, $s \in \{1, 2, 3\}$	$h_i(x_s) = 0$, $h_j(x_s) = 0$, $e_i(x_s) = 0$, $e_j(x_s) = 0$, $f_i(x_s) = 0$, $f_j(x_s) = 0$, $s \in \{1, 2, 3\}$
$k_i(x_1) = q^{-2}x_1$, $k_i(x_2) = q^{-1}x_2$, $k_i(x_3) = q^{-1}x_3$, $k_j(x_1) = qx_1$, $k_j(x_2) = qx_2$, $k_j(x_3) = q^2x_3$, $e_i(x_1) = 1$, $e_i(x_2) = 0$, $e_i(x_3) = 0$, $e_j(x_1) = -qx_1x_3$, $e_j(x_2) = -qx_2x_3$, $e_j(x_3) = -qx_3^2$, $f_i(x_1) = -qx_1^2$, $f_i(x_2) = -qx_1x_2$, $f_i(x_3) = -qx_1x_3$, $f_j(x_1) = 0$, $f_j(x_2) = 0$, $f_j(x_3) = 1$	$h_i(x_1) = -2x_1$, $h_i(x_2) = -x_2$, $h_i(x_3) = -x_3$, $h_j(x_1) = x_1$, $h_j(x_2) = x_2$, $h_j(x_3) = 2x_3$, $e_i(x_1) = 1$, $e_i(x_2) = 0$, $e_i(x_3) = 0$, $e_j(x_1) = -x_1x_3$, $e_j(x_2) = -x_2x_3$, $e_j(x_3) = -x_3^2$, $f_i(x_1) = -x_1^2$, $f_i(x_2) = -x_1x_2$, $f_i(x_3) = -x_1x_3$, $f_j(x_1) = 0$, $f_j(x_2) = 0$, $f_j(x_3) = 1$
$k_i(x_1) = q^{-2}x_1$, $k_i(x_2) = q^{-1}x_2$, $k_i(x_3) = q^{-1}x_3$, $k_j(x_1) = qx_1$, $k_j(x_2) = q^{-1}x_2$, $k_j(x_3) = x_3$, $e_i(x_1) = 1$, $e_i(x_2) = 0$, $e_i(x_3) = 0$, $e_j(x_1) = 0$, $e_j(x_2) = x_1$, $e_j(x_3) = 0$, $f_i(x_1) = -qx_1^2$, $f_i(x_2) = -qx_1x_2$, $f_i(x_3) = -qx_1x_3$, $f_j(x_1) = x_2$, $f_j(x_2) = 0$, $f_j(x_3) = 0$	$h_i(x_1) = -2x_1$, $h_i(x_2) = -x_2$, $h_i(x_3) = -x_3$, $h_j(x_1) = x_1$, $h_j(x_2) = -x_2$, $h_j(x_3) = 0$, $e_i(x_1) = 1$, $e_i(x_2) = 0$, $e_i(x_3) = 0$, $e_j(x_1) = 0$, $e_j(x_2) = x_1$, $e_j(x_3) = 0$, $f_i(x_1) = -x_1^2$, $f_i(x_2) = -x_1x_2$, $f_i(x_3) = -x_1x_3$, $f_j(x_1) = x_2$, $f_j(x_2) = 0$, $f_j(x_3) = 0$

$k_i(x_1) = qx_1,\ k_i(x_2) = qx_2,$ $k_i(x_3) = q^2x_3,\ k_j(x_1) = x_1,$ $k_j(x_2) = qx_2,\ k_j(x_3) = q^{-1}x_3,$ $e_i(x_1) = -qx_1x_3,\ e_i(x_2) = -qx_2x_3,$ $e_i(x_3) = -qx_3^2,\ e_j(x_1) = 0,$ $e_j(x_2) = 0,\ e_j(x_3) = x_2,$ $f_i(x_1) = 0,\ f_i(x_2) = 0,\ f_i(x_3) = 1,$ $f_j(x_1) = 0,\ f_j(x_2) = x_3,\ f_j(x_3) = 0$	$h_i(x_1) = x_1,\ h_i(x_2) = x_2,$ $h_i(x_3) = 2x_3,\ h_j(x_1) = 0,$ $h_j(x_2) = x_2,\ h_j(x_3) = -x_3,$ $e_i(x_1) = -x_1x_3,\ e_i(x_2) = -x_2x_3,$ $e_i(x_3) = -x_3^2,\ e_j(x_1) = 0,$ $e_j(x_2) = 0,\ e_j(x_3) = x_2,$ $f_i(x_1) = 0,\ f_i(x_2) = 0,\ f_i(x_3) = 1,$ $f_j(x_1) = 0,\ f_j(x_2) = x_3,\ f_j(x_3) = 0$
$k_i(x_1) = qx_1,\ k_i(x_2) = q^{-1}x_2,$ $k_i(x_3) = x_3,\ k_j(x_1) = x_1,$ $k_j(x_2) = qx_2,\ k_j(x_3) = q^{-1}x_3,$ $e_i(x_1) = 0,\ e_i(x_2) = x_1,\ e_i(x_3) = 0,$ $e_j(x_1) = 0,\ e_j(x_2) = 0,\ e_j(x_3) = x_2,$ $f_i(x_1) = x_2,\ f_i(x_2) = 0,\ f_i(x_3) = 0,$ $f_j(x_1) = 0,\ f_j(x_2) = x_3,\ f_j(x_3) = 0$	$h_i(x_1) = x_1,\ h_i(x_2) = -x_2,$ $h_i(x_3) = 0,\ h_j(x_1) = 0,$ $h_j(x_2) = x_2,\ h_j(x_3) = -x_3,$ $e_i(x_1) = 0,\ e_i(x_2) = x_1,\ e_i(x_3) = 0,$ $e_j(x_1) = 0,\ e_j(x_2) = 0,\ e_j(x_3) = x_2,$ $f_i(x_1) = x_2,\ f_i(x_2) = 0,\ f_i(x_3) = 0,$ $f_j(x_1) = 0,\ f_j(x_2) = x_3,\ f_j(x_3) = 0$
$k_i(x_1) = qx_1,\ k_i(x_2) = q^{-1}x_2,$ $k_i(x_3) = x_3,\ k_j(x_1) = qx_1,$ $k_j(x_2) = q^2x_2,\ k_j(x_3) = q^{-1}x_3,$ $e_i(x_1) = 0,\ e_i(x_2) = x_1,\ e_i(x_3) = 0,$ $e_j(x_1) = -qx_1x_2,\ e_j(x_2) = -qx_2^2,$ $e_j(x_3) = x_2x_3,\ f_i(x_1) = x_2,$ $f_i(x_2) = 0,\ f_i(x_3) = 0,$ $f_j(x_1) = 0,\ f_j(x_2) = 1,\ f_j(x_3) = 0$	$h_i(x_1) = x_1,\ h_i(x_2) = -x_2,$ $h_i(x_3) = 0,\ h_j(x_1) = x_1,$ $h_j(x_2) = 2x_2,\ h_j(x_3) = -x_3,$ $e_i(x_1) = 0,\ e_i(x_2) = x_1,\ e_i(x_3) = 0,$ $e_j(x_1) = -x_1x_2,\ e_j(x_2) = -x_2^2,$ $e_j(x_3) = x_2x_3,\ f_i(x_1) = x_2,$ $f_i(x_2) = 0,\ f_i(x_3) = 0,$ $f_j(x_1) = 0,\ f_j(x_2) = 1,\ f_j(x_3) = 0$
$k_i(x_1) = x_1,\ k_i(x_2) = qx_2,$ $k_i(x_3) = q^{-1}x_3,\ k_j(x_1) = qx_1,$ $k_j(x_2) = q^{-2}x_2, k_j(x_3) = q^{-1}x_3,$ $e_i(x_1) = 0,\ e_i(x_2) = 0,$ $e_i(x_3) = x_2,\ e_j(x_1) = 0,$ $e_j(x_2) = 1,\ e_j(x_3) = 0,$ $f_i(x_1) = 0,\ f_i(x_2) = x_3,\ f_i(x_3) = 0,$ $f_j(x_1) = x_1x_2,\ f_j(x_2) = -qx_2^2,$ $f_j(x_3) = -qx_2x_3$	$h_i(x_1) = 0,\ h_i(x_2) = x_2,$ $h_i(x_3) = -x_3,\ h_j(x_1) = x_1,$ $h_j(x_2) = -2x_2,\ h_j(x_3) = -x_3,$ $e_i(x_1) = 0,\ e_i(x_2) = 0,$ $e_i(x_3) = x_2,\ e_j(x_1) = 0,$ $e_j(x_2) = 1,\ e_j(x_3) = 0,$ $f_i(x_1) = 0,\ f_i(x_2) = x_3,\ f_i(x_3) = 0,$ $f_j(x_1) = x_1x_2,\ f_j(x_2) = -x_2^2,$ $f_j(x_3) = -x_2x_3$
$k_i(x_1) = q^{-2}x_1,\ k_i(x_2) = q^{-1}x_2,$ $k_i(x_3) = q^{-1}x_3,\ k_j(x_1) = qx_1,$ $k_j(x_2) = q^2x_2,\ k_j(x_3) = q^{-1}x_3,$ $e_i(x_1) = 1,\ e_i(x_2) = 0,\ e_i(x_3) = 0,$ $e_j(x_1) = -qx_1x_2,\ e_j(x_2) = -qx_2^2,$ $e_j(x_3) = x_1 + x_2x_3,\ f_i(x_1) = -qx_1^2,$ $f_i(x_2) = -qx_1x_2,$ $f_i(x_3) = -qx_1x_3 - qx_2x_3^2,$ $f_j(x_1) = 0,\ f_j(x_2) = 1,\ f_j(x_3) = 0$	$h_i(x_1) = -2x_1,\ h_i(x_2) = -x_2,$ $h_i(x_3) = -x_3,\ h_j(x_1) = x_1,$ $h_j(x_2) = 2x_2,\ h_j(x_3) = -x_3,$ $e_i(x_1) = 1,\ e_i(x_2) = 0,\ e_i(x_3) = 0,$ $e_j(x_1) = -x_1x_2,\ e_j(x_2) = -x_2^2,$ $e_j(x_3) = x_1 + x_2x_3,\ f_i(x_1) = -x_1^2,$ $f_i(x_2) = -x_1x_2,$ $f_i(x_3) = -x_1x_3 - x_2x_3^2,$ $f_j(x_1) = 0,\ f_j(x_2) = 1,\ f_j(x_3) = 0$
$k_i(x_1) = qx_1,\ k_i(x_2) = qx_2,$ $k_i(x_3) = q^2x_3,\ k_j(x_1) = qx_1,$ $k_j(x_2) = q^{-2}x_2,\ k_j(x_3) = q^{-1}x_3,$ $e_i(x_1) = -qx_1x_3 - qx_1^2x_2,$ $e_i(x_2) = -qx_2x_3,\ e_i(x_3) = -qx_3^2,$ $e_j(x_1) = 0,\ e_j(x_2) = 1,\ e_j(x_3) = 0,$ $f_i(x_1) = 0,\ f_i(x_2) = 0,\ f_i(x_3) = 1,$ $f_j(x_1) = x_3 + x_1x_2,\ f_j(x_2) = -qx_2^2,$ $f_j(x_3) = -qx_2x_3$	$h_i(x_1) = x_1,\ h_i(x_2) = x_2,$ $h_i(x_3) = 2x_3,\ h_j(x_1) = x_1,$ $h_j(x_2) = -2x_2,\ h_j(x_3) = -x_3,$ $e_i(x_1) = -x_1x_3 - x_1^2x_2,$ $e_i(x_2) = -yz,\ e_i(x_3) = -x_3^2,$ $e_j(x_1) = 0,\ e_j(x_2) = 1,\ e_j(x_3) = 0,$ $f_i(x_1) = 0,\ f_i(x_2) = 0,\ f_i(x_3) = 1,$ $f_j(x_1) = x_3 + x_1x_2,\ f_j(x_2) = -x_2^2,$ $f_j(x_3) = -x_2x_3$

for any $i = 1$, $j = 2$ or $i = 2$, $j = 1$. Moreover, there are no isomorphisms between these nine kinds of module-algebra structures.

Remark 5.2. Case (5) when $i = 1$, $j = 2$ in Theorem 5.1 is the case discussed in [15] when $n = 3$.

Let us denote the actions of $U_q(sl(2))$ on $A_q(3)$ in (A1), those in (A2) and (B3) in Lemma 4.11, those in (B4) in Lemma 4.12, those in (B5) in Lemma 4.14, those in (B6) in Lemma 4.17 and those in (B7) in Lemma 4.18 by $\star 1$, $\star 2$, $\star 3$, $\star 4$, $\star 5$, $\star 6$, $\star 7$ respectively. In addition, denote the actions of $U_q(sl(2))$ on $A_q(3)$ in (A2) and (B7) in Lemma 4.13, those in (A3) and (B6) in Lemma 4.15 by $\star 2'$, $\star 7'$, $\star 3'$, $\star 6'$ respectively. If $\star s$ and $\star t$ are compatible, in other words, they determine a $U_q(sl(3))$-module algebra structure on $A_q(3)$, we use an edge connecting $\star s$ and $\star t$, since k_1, e_1, f_1 and k_2, e_2, f_2 are symmetric in $U_q(sl(3))$. Then, we can use the following diagrams to denote all actions of $U_q(sl(3)) = \mathcal{H}(e_i, f_i, k_i, k_i^{-1})_{i=1,2}$ on $A_q(3)$ when $k_i \in \mathrm{Aut}\ \mathrm{L}(A_q(3))$ for $i = 1, 2$:

$$\star 1 \;\text{——}\; \star 1\,, \quad \star 7' \;\text{——}\; \star 2'\,, \quad \star 3' \;\text{——}\; \star 6'\,, \tag{5.65}$$

$$\begin{tikzcd}[arrows=dash] & \star 2 \arrow[r] \arrow[d] & \star 3 \arrow[d] & \\ \star 7 \arrow[r] & \star 4 \arrow[r] & \star 5 \arrow[r] & \star 6 \end{tikzcd} \quad . \tag{5.66}$$

Here, every two adjacent vertices corresponds to two classes of the module-algebra structures of $U_q(sl(3))$ on $A_q(3)$. For example, $\star 2 \text{——} \star 3$ corresponds to the following two kinds of module-algebra structures of $U_q(sl(3))$ on $A_q(3)$: one has actions of k_1, e_1, f_1 that are of type $\star 2$ and actions of k_2, e_2, f_2 that are of type $\star 3$; the other has actions of k_1, e_1, f_1 that are of type $\star 3$ and actions of k_2, e_2, f_2 that are of type $\star 2$.

Next, we will begin to study the module-algebra structures of $U_q(sl(m+1)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{1 \le i \le m}$ on $A_q(3)$, when $k_i \in \mathrm{Aut}\ \mathrm{L}(A_q(3))$ for $i = 1, \cdots, m$ and $m \ge 3$. The corresponding Dynkin diagram of $sl(m+1)$ with m vertices is as follows:

$$\circ \text{——} \circ \text{——} \cdots \text{——} \circ \text{——} \circ\,.$$

In $U_q(sl(m+1))$, every vertex corresponds to one Hopf subalgebra isomorphic to $U_q(sl(2))$ and two adjacent vertices correspond to one Hopf subalgebra isomorphic to $U_q(sl(3))$. Therefore, for studying the module-algebra structures of $U_q(sl(m+1))$ on $A_q(3)$, we have to endow every vertex in the Dynkin diagram of $sl(m+1)$ an action of $U_q(sl(2))$ on $A_q(3)$. Moreover, there are some rules which we should obey:

1. Since every pair of adjacent vertices in the Dynkin diagram corresponds to one Hopf subalgebra isomorphic to $U_q(sl(3))$, by Theorem 5.1, the action of $U_q(sl(2))$ on $A_q(3)$ on every vertex should be one of the following 11 kinds of possibilities: $\star 1$, $\star 2$, $\star 3$, $\star 4$, $\star 5$, $\star 6$, $\star 7$, $\star 2'$, $\star 7'$, $\star 3'$, $\star 6'$. Moreover, every pair of adjacent vertices should be of the types in (5.65) and (5.66).

2. Except for $\star 1$, any type of action of $U_q(sl(2))$ on $A_q(3)$ cannot be endowed with two different vertices simultaneously, since the relations (2.2) acting on x_1, x_2, x_3 to produce zero cannot be satisfied.

3. If every vertex in the Dynkin diagram of $sl(m+1)$ is endowed with an action of $U_q(sl(2))$ on $A_q(3)$ which is not of Case $\star 1$, any two vertices which are not adjacent cannot be endowed with the types which are adjacent (5.65) and (5.66).

Theorem 5.3. *If $m \geq 4$, all module-algebra structures of $U_q(sl(m+1)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{1\leq i\leq m}$ on $A_q(3)$ when $k_i \in Aut\ L(A_q(3))$ for $i = 1, \cdots, m$ are as follows*

$$
\begin{aligned}
&k_i(x_1) = \pm x_1, \qquad k_i(x_2) = \pm x_2, \qquad k_i(x_3) = \pm x_3,\\
&e_i(x_1) = e_i(x_2) = e_i(x_3) = f_i(x_1) = f_i(x_2) = f_i(x_3) = 0,
\end{aligned}
$$

for any $i \in \{1, 2, \cdots, m\}$.

For $m = 3$, all module-algebra structures of $U_q(sl(4)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{i=1,2,3}$ on $A_q(3)$ when $k_i \in Aut\ L(A_q(3))$ for $i = 1, 2, 3$ are given by
(1)

$$
\begin{aligned}
&k_i(x_1) = \pm x_1, \qquad k_i(x_2) = \pm x_2, \qquad k_i(x_3) = \pm x_3,\\
&e_i(x_1) = e_i(x_2) = e_i(x_3) = f_i(x_1) = f_i(x_2) = f_i(x_3) = 0,
\end{aligned}
$$

for any $i \in \{1, 2, 3\}$. All these module-algebra structures are not pairwise non-isomorphic.
(2)

$$
\begin{aligned}
&k_i(x_1) = qx_1, \quad k_i(x_2) = q^{-1}x_2, \quad k_i(x_3) = x_3,\\
&e_i(x_1) = 0, \quad e_i(x_2) = c_{21}x_1, \quad e_i(x_3) = 0,\\
&f_i(x_1) = c_{21}^{-1}x_2, \quad f_i(x_2) = 0, \quad f_i(x_3) = 0,\\
&k_j(x_1) = q^{-2}x_1, \quad k_j(x_2) = q^{-1}x_2, \quad k_j(x_3) = q^{-1}x_3,\\
&e_j(x_1) = a_1, \quad e_j(x_2) = 0, \quad e_j(x_3) = 0,\\
&f_j(x_1) = -qa_1^{-1}x_1^2, \quad f_j(x_2) = -qa_1^{-1}x_1x_2, \quad f_j(x_3) = -qa_1^{-1}x_1x_3,\\
&k_s(x_1) = qx_1, \quad k_s(x_2) = qx_2, k_s(x_3) = q^2x_3,\\
&e_s(x_1) = -qb_3^{-1}x_1x_3, \quad e_s(x_2) = -qb_3^{-1}x_2x_3, \quad e_s(x_3) = -qb_3^{-1}x_3^2,\\
&f_s(x_1) = 0, \quad f_s(x_2) = 0, \quad f_s(x_3) = b_3,
\end{aligned}
$$

where a_1, b_3, $c_{21} \in \mathbb{C}\backslash\{0\}$ and $i = 1$, $j = 2$, $s = 3$ or $i = 3$, $j = 2$, $s = 1$. All these module-algebra structures are isomorphic to that with $a_1 = b_3 = c_{21} = 1$.

(3)

$$\begin{aligned}
&k_i(x_1) = qx_1, \quad k_i(x_2) = q^{-1}x_2, \quad k_i(x_3) = x_3,\\
&e_i(x_1) = 0, \quad e_i(x_2) = c_{21}x_1, \quad e_i(x_3) = 0,\\
&f_i(x_1) = c_{21}^{-1}x_2, \quad f_i(x_2) = 0, \quad f_i(x_3) = 0,\\
&k_j(x_1) = x_1, \quad k_j(x_2) = qx_2, \quad k_j(x_3) = q^{-1}x_3,\\
&e_j(x_1) = 0, \quad e_j(x_2) = 0, \quad e_j(x_3) = c_{32}x_2,\\
&f_j(x_1) = 0, \quad f_j(x_2) = c_{32}^{-1}x_3, \quad f_j(x_3) = 0,\\
&k_s(x_1) = qx_1, \quad k_s(x_2) = qx_2, \quad k_s(x_3) = q^2x_3,\\
&e_s(x_1) = -qb_3^{-1}x_1x_3, \quad e_s(x_2) = -qb_3^{-1}x_2x_3, \quad e_s(x_3) = -qb_3^{-1}x_3^2,\\
&f_s(x_1) = 0, \quad f_s(x_2) = 0, \quad f_s(x_3) = b_3,
\end{aligned}$$

where b_3, c_{21}, $c_{32} \in \mathbb{C}\backslash\{0\}$ and $i = 1$, $j = 2$, $s = 3$ or $i = 3$, $j = 2$, $s = 1$. All these module-algebra structures are isomorphic to that with $b_3 = c_{21} = c_{32} = 1$.
(4)

$$\begin{aligned}
&k_i(x_1) = q^{-2}x_1, \quad k_i(x_2) = q^{-1}x_2, \quad k_i(x_3) = q^{-1}x_3,\\
&e_i(x_1) = a_1, \quad e_i(x_2) = 0, \quad e_i(x_3) = 0,\\
&f_i(x_1) = -qa_1^{-1}x_1^2, \quad f_i(x_2) = -qa_1^{-1}x_1x_2, \quad f_i(x_3) = -qa_1^{-1}x_1x_3,\\
&k_j(x_1) = qx_1, \quad k_j(x_2) = qx_2, \quad k_j(x_3) = q^2x_3,\\
&e_j(x_1) = -qb_3^{-1}x_1x_3, \quad e_j(x_2) = -qb_3^{-1}x_2x_3, \quad e_j(x_3) = -qb_3^{-1}x_3^2,\\
&f_j(x_1) = 0, \quad f_j(x_2) = 0, \quad f_j(x_3) = b_3,\\
&k_s(x_1) = x_1, \quad k_s(x_2) = qx_2, \quad k_s(x_3) = q^{-1}x_3,\\
&e_s(x_1) = 0, \quad e_s(x_2) = 0, \quad e_s(x_3) = c_{32}x_2,\\
&f_s(x_1) = 0, \quad f_s(x_2) = c_{32}^{-1}x_3, \quad f_s(x_3) = 0,
\end{aligned}$$

where a_1, b_3, $c_{32} \in \mathbb{C}\backslash\{0\}$ and $i = 1$, $j = 2$, $s = 3$ or $i = 3$, $j = 2$, $s = 1$. All these module-algebra structures are isomorphic to that with $a_1 = b_3 = c_{32} = 1$.
(5)

$$\begin{aligned}
&k_i(x_1) = q^{-2}x_1, \quad k_i(x_2) = q^{-1}x_2, \quad k_i(x_3) = q^{-1}x_3,\\
&e_i(x_1) = a_1, \quad e_i(x_2) = 0, \quad e_i(x_3) = 0,\\
&f_i(x_1) = -qa_1^{-1}x_1^2, \quad f_i(x_2) = -qa_1^{-1}x_1x_2, \quad f_i(x_3) = -qa_1^{-1}x_1x_3,\\
&k_j(x_1) = qx_1, \quad k_j(x_2) = q^{-1}x_2, \quad k_j(x_3) = x_3,\\
&e_j(x_1) = 0, \quad e_j(x_2) = c_{21}x_1, \quad e_j(x_3) = 0,\\
&f_j(x_1) = c_{21}^{-1}x_2, \quad f_j(x_2) = 0, \quad f_j(x_3) = 0,\\
&k_s(x_1) = x_1, \quad k_s(x_2) = qx_2, \quad k_s(x_3) = q^{-1}x_3,\\
&e_s(x_1) = 0, \quad e_s(x_2) = 0, \quad e_s(x_3) = c_{32}x_2,\\
&f_s(x_1) = 0, \quad f_s(x_2) = c_{32}^{-1}x_3, \quad f_s(x_3) = 0,
\end{aligned}$$

where a_1, c_{21}, $c_{32} \in \mathbb{C}\backslash\{0\}$ and $i = 1$, $j = 2$, $s = 3$ or $i = 3$, $j = 2$, $s = 1$. All these module-algebra structures are isomorphic to that with $a_1 = c_{21} = c_{32} = 1$.

Proof. First, we consider the case when $m \geq 5$. By the above discussion, since there are no paths in (5.66) whose length is larger than 4 and any two vertices which are not adjacent in this path have no edge connecting them in (5.65) and (5.66), the unique possibility of putting the actions of $U_q(sl(2))$ on the m vertices in the Dynkin diagram is as follows:

$$\star 1 \text{ --- } \star 1 \text{ --- } \cdots \text{ --- } \star 1 \text{ --- } \star 1 \ .$$

Obviously, the above case determines the module-algebra structures of $U_q(sl(m+1))$ on $A_q(3)$.

Second, let us study the case when $m = 3$. By the above rules, and because the Dynkin diagram of $sl(4)$ is symmetric, we only need to check the following cases

$$\begin{array}{lll} \star 7 \text{ --- } \star 4 \text{ --- } \star 5\ , & \star 7 \text{ --- } \star 4 \text{ --- } \star 2\ , & \star 4 \text{ --- } \star 2 \text{ --- } \star 3\ , \\ \star 4 \text{ --- } \star 5 \text{ --- } \star 3\ , & \star 4 \text{ --- } \star 5 \text{ --- } \star 6\ , & \star 2 \text{ --- } \star 3 \text{ --- } \star 5\ , \\ \star 2 \text{ --- } \star 4 \text{ --- } \star 5\ , & \star 3 \text{ --- } \star 5 \text{ --- } \star 6\ , & \star 1 \text{ --- } \star 1 \text{ --- } \star 1\ . \end{array}$$

To determine the module-algebra structures of $U_q(sl(4))$ on $A_q(3)$, we still have to check the following equalities

$$\begin{array}{llll} k_1e_3(u) = e_3k_1(u), & k_1f_3(u) = f_3k_1(u), & k_3e_1(u) = e_1k_3(u), & k_3f_1(u) = f_1k_3(u), \\ e_1f_3(u) = f_3e_1(u), & e_3f_1(u) = f_1e_3(u), & e_1e_3(u) = e_3e_1(u), & f_1f_3(u) = f_3f_1(u), \end{array}$$

for any $u \in \{x_1, x_2, x_3\}$. For $\star 7 \text{ --- } \star 4 \text{ --- } \star 5$, since $k_1e_3(x_3) = k_1(b_3x_2) = q^2b_3x_2$ and $e_3k_1(x_3) = q^{-1}b_3x_2$, $k_1e_3(x_3) \neq e_3k_1(x_3)$. Therefore, $\star 7 \text{ --- } \star 4 \text{ --- } \star 5$ is excluded. Similarly, we exclude $\star 7 \text{ --- } \star 4 \text{ --- } \star 2$, $\star 4 \text{ --- } \star 5 \text{ --- } \star 6$, and $\star 3 \text{ --- } \star 5 \text{ --- } \star 6$. Moreover, it is easy to check the five remaining cases determine the module-algebra structures of $U_q(sl(4))$ on $A_q(3)$.

Thirdly, we consider the case when $m = 4$. By the discussion above, we only need to check the cases

$$\begin{array}{ll} \star 7 \text{ --- } \star 4 \text{ --- } \star 2 \text{ --- } \star 3\ , & \star 7 \text{ --- } \star 4 \text{ --- } \star 5 \text{ --- } \star 3\ , \\ \star 7 \text{ --- } \star 4 \text{ --- } \star 5 \text{ --- } \star 6\ , & \star 2 \text{ --- } \star 3 \text{ --- } \star 5 \text{ --- } \star 6\ , \\ \star 1 \text{ --- } \star 1 \text{ --- } \star 1 \text{ --- } \star 1\ . & \end{array}$$

Since the three adjacent vertices in the Dynkin diagram of $sl(5)$ correspond to one Hopf algebra isomorphic to $U_q(sl(4))$, by the results of the module-algebra structures of $U_q(sl(4))$ on $A_q(3)$, there is only one possibility:

$$\star 1 \text{ --- } \star 1 \text{ --- } \star 1 \text{ --- } \star 1\ .$$

Finally, we consider the isomorphism classes. Here, we will only show that all the module-algebra structures of $U_q(sl(4))$ on $A_q(3)$ in Case (2) are isomorphic to that with $a_1 = c_{21} = b_3 = 1$. The desired isomorphism is given by $\psi_{a_1,c_{21},b_3} : x_1 \to a_1x_1, x_2 \to a_1c_{21}x_2, x_3 \to b_3x_3$. The other cases can be considered similarly. ■

Remark 5.4. By Theorem 5.3, the classical limits of the above actions, i.e. the Lie algebra sl_{m+1}-actions by differentiations on $\mathbb{C}[x_1, x_2, x_3]$ can also be obtained, as before.

Finally, we present a classification of $U_q(sl(m+1))$-module algebra structures on $A_q(2)$.

Since all module-algebra structures of $U_q(sl(2))$ on $A_q(2)$ are presented in [8], using the same method as above and by some computations, we can obtain the following theorem.

Theorem 5.5. *$U_q(sl(3))$-module algebra structures on $A_q(2)$ up to isomorphisms and their classical limits are as follows:*

$U_q(sl(3))$-*module* *algebra structures*	*Classical limit* sl_3-*actions on* $\mathbb{C}[x_1, x_2]$
$k_i(x_s) = \pm x_s,\ k_j(x_s) = \pm x_s,$ $e_i(x_s) = 0,\ e_j(x_s) = 0,$ $f_i(x_s) = 0,\ f_j(x_s) = 0,$ $s \in \{1, 2\}$	$h_i(x_s) = 0,\ h_j(x_s) = 0,$ $e_i(x_s) = 0,\ e_j(x_s) = 0,$ $f_i(x_s) = 0,\ f_j(x_s) = 0,$ $s \in \{1, 2\}$
$k_i(x_1) = q^{-2}x_1,\ k_i(x_2) = q^{-1}x_2,$ $k_j(x_1) = qx_1,\ k_j(x_2) = q^{-1}x_2,$ $e_i(x_1) = 1,\ e_i(x_2) = 0,$ $e_j(x_1) = 0,\ e_j(x_2) = x_1,$ $f_i(x_1) = -qx_1^2,\ f_i(x_2) = -qx_1x_2,$ $f_j(x_1) = x_2,\ f_j(x_2) = 0$	$h_i(x_1) = -2x_1,\ h_i(x_2) = -x_2,$ $h_j(x_1) = x_1,\ h_j(x_2) = -x_2,$ $e_i(x_1) = 1,\ e_i(x_2) = 0,$ $e_j(x_1) = 0,\ e_j(x_2) = x_1,$ $f_i(x_1) = -x^2,\ f_i(x_2) = -x_1x_2,$ $f_j(x_1) = x_2,\ f_j(x_2) = 0$
$k_i(x_1) = qx_1,\ k_i(x_2) = q^2x_2,$ $k_j(x_1) = qx_1,\ k_j(x_2) = q^{-1}x_2,$ $e_i(x_1) = -qx_1x_2,\ e_i(x_2) = -qx_2^2,$ $e_j(x_1) = 0,\ e_j(x_2) = x_1,$ $f_i(x_1) = 0,\ f_i(x_2) = 1,$ $f_j(x_1) = x_2,\ f_j(x_2) = 0$	$h_i(x_1) = x_1,\ h_i(x_2) = 2x_2,$ $h_j(x_1) = x_1,\ h_j(x_2) = -x_2,$ $e_i(x_1) = -x_1x_2,\ e_i(x_2) = -x_2^2,$ $e_j(x_1) = 0,\ e_j(x_2) = x_1,$ $f_i(x_1) = 0,\ f_i(x_2) = 1,$ $f_j(x_1) = x_2,\ f_j(x_2) = 0$
$k_i(x_1) = q^{-2}x_1,\ k_i(x_2) = q^{-1}x_2,$ $k_j(x_1) = qx_1,\ k_j(x_2) = q^2x_2,$ $e_i(x_1) = 1,\ e_i(x_2) = 0,$ $e_j(x_1) = -qx_1x_2,\ e_j(x_2) = -qx_2^2,$ $f_i(x_1) = -qx_1^2,\ f_i(x_2) = -qx_1x_2,$ $f_j(x_1) = 0,\ f_j(x_2) = 1$	$h_i(x_1) = -2x_1,\ h_i(x_2) = -x_2,$ $h_j(x_1) = x_1,\ h_j(x_2) = 2x_2,$ $e_i(x_1) = 1,\ e_i(x_2) = 0,$ $e_j(x_1) = -x_1x_2,\ e_j(x_2) = -x_2^2,$ $f_i(x_1) = -x_1^2,\ f_i(x_2) = -x_1x_2,$ $f_j(x_1) = 0,\ f_j(x_2) = 1$

for any $i = 1,\ j = 2$ or $i = 2,\ j = 1$. Note that there are no isomorphisms between these four kinds of module-algebra structures.

Moreover, for any $m \geq 3$, all module-algebra structures of $U_q(sl(m+1))$ on $A_q(2)$ are as follows:

$$k_i(x_1) = \pm x_1, \qquad k_i(x_2) = \pm x_2,$$
$$e_i(x_1) = e_i(x_2) = f_i(x_1) = f_i(x_2) = 0,$$

for any $i \in \{1, \cdots, m\}$.

6. $U_q(sl(m+1))$-module algebra structures on $A_q(n)$ $(n \geq 4)$

In this section, we will study the module algebra structures of $U_q(sl(m+1)) = \mathcal{H}(e_i, f_i, k_i^{\pm 1})_{1\leq i\leq m}$ on $A_q(n)$ when $k_i \in AutL(A_q(n))$ for $i = 1, \cdots, m$ and $n \geq 4$.

By Theorem 3.1 and with a similar discussion in Section 4, we can obtain the following proposition.

Proposition 6.1. *For $n \geq 4$, the module-algebra structures of $U_q(sl(2))$ on $A_q(n)$ are as follows:*

(1) $$k(x_i) = \pm x_i, \quad e(x_i) = f(x_i) = 0,$$

for any $i \in \{1, \cdots, n\}$. All these structures are pairwise nonisomorphic.

(2) $$\begin{aligned} & k(x_i) = qx_i \ \text{ for } \forall \ i < j, \ \ k(x_j) = q^{-2}x_j, \ \ k(x_i) = q^{-1}x_i \ \text{ for } \forall \ i > j, \\ & e(x_i) = 0 \ \text{ for } \forall \ i \neq j, \ \ e(x_j) = a_j, \\ & f(x_i) = a_j^{-1}x_ix_j \ \text{ for } \forall \ i < j, \ \ f(x_j) = -qa_j^{-1}x_j^2, \\ & f(x_i) = -qa_j^{-1}x_jx_i \quad \text{ for } \forall \ i > j, \end{aligned}$$

for any $j \in \{1, \cdots, n\}$ and $a_j \in \mathbb{C} \setminus \{0\}$. If j is fixed, then all of these structures are isomorphic to that with $a_j = 1$.

(3) $$\begin{aligned} & k(x_i) = qx_i \ \text{ for } \forall \ i < j, \ \ k(x_j) = q^2x_j, \ \ k(x_i) = q^{-1}x_i \ \text{ for } \forall \ i > j, \\ & e(x_i) = -qb_j^{-1}x_ix_j \ \text{ for } \forall \ i < j, \ \ e(x_j) = -qb_j^{-1}x_j^2, \\ & e(x_i) = b_j^{-1}x_jx_i \quad \text{ for } \forall \ i > j, \qquad f(x_i) = 0 \ \text{ for } \forall \ i \neq j, \ \ f(x_j) = b_j, \end{aligned}$$

for any $j \in \{1, \cdots, n\}$ and $b_j \in \mathbb{C} \setminus \{0\}$. If j is fixed, then all of these structures are isomorphic to that with $b_j = 1$.

(4) $$\begin{aligned} & k(x_i) = x_i \ \text{ for } \forall \ i < j, \ \ k(x_j) = qx_j, \\ & k(x_{j+1}) = q^{-1}x_{j+1}, \ \ k(x_i) = x_i \ \text{ for } \forall \ i > j+1, \\ & e(x_i) = 0 \ \text{ for } \forall \ i \neq j+1, \ \ e(x_{j+1}) = c_{j+1,j}x_j, \\ & f(x_i) = 0 \ \text{ for } \forall \ i \neq j, \ \ f(x_j) = c_{j+1,j}^{-1}x_{j+1}, \end{aligned}$$

for any $j \in \{1, \cdots, n-1\}$ and $c_{j+1,j} \in \mathbb{C} \setminus \{0\}$. If j is fixed, then all of these structures are isomorphic to that with $c_{j+1,j} = 1$.

Remark 6.2. In Proposition 6.1 we have presented only the simplest module-algebra structures. It is also complicated to give the solutions of (3.18) and (3.19) for all cases. For example, by a very complex computation, we can obtain that in Case $\left(\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_0, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_1 \right)$, all $U_q(sl(2))$-module algebra structures on $A_q(n)$ are given by

$$\begin{aligned}
k(x_1) &= q^{-2}x_1, \quad k(x_i) = q^{-1}x_i \ \text{ for } \ \forall \ \ i>1,\\
e(x_1) &= a_1, \quad e(x_i) = 0 \ \text{ for } \ \forall \ \ i>1,\\
f(x_1) &= -qa_1^{-1}x_1^2,\\
f(x_2) &= -qa_1^{-1}x_1x_2 + \sum_{2<s\le n}\widehat{v}_{2s2}x_2x_s^2 + \sum_{2<s<k<l\le n}\alpha_{22kl}x_2x_kx_l + \beta_{22}x_2^3,\\
f(x_i) &= -qa_1^{-1}x_1x_i + (3)_q\beta_{22}x_2^2x_i - \sum_{2<s<i\le n} q^{-1}(3)_q\widehat{v_{2s2}}x_s^2x_i\\
&\quad +\frac{(2)_q}{(3)_q}\alpha_{n2in}x_2x_i^2 + \sum_{2<i<t\le n}\widehat{v_{2t2}}x_ix_t^2 - \sum_{2<s<i<n} q^{-1}(2)_q\alpha_{22si}\\
&\quad \cdot x_sx_i^2 + \frac{q}{(2)_q}\widehat{v_{nn2}}x_2x_ix_n + \sum_{2<k<i<n}\alpha_{n2kn}x_2x_kx_i\\
&\quad + \sum_{2<i<k<n}\frac{q}{(3)_q}\alpha_{n2kn}x_2x_ix_k - \sum_{2<s<i<k\le n}\alpha_{22sk}x_sx_ix_k\\
&\quad + \sum_{2<i<k<l\le n}\alpha_{22kl}x_ix_kx_l - \sum_{2<s<k<i<n} q^{-1}(3)_q\alpha_{22sk}x_sx_kx_i\\
&\quad -q^{-1}\widehat{v_{2i2}}x_i^3,
\end{aligned}$$

where $2<i<n$,

$$\begin{aligned}
f(x_n) &= -qa_1^{-1}x_1x_n + (3)_q\beta_{22}x_2^2x_n - \sum_{2<s<n} q^{-1}(3)_q\widehat{v_{2s2}}x_s^2x_n\\
&\quad +\widehat{v_{nn2}}x_2x_n^2 - \sum_{2<k<n} q^{-1}(2)_q\alpha_{22kn}x_kx_n^2 + \sum_{2<k<n}\alpha_{n2kn}x_2x_kx_n\\
&\quad - \sum_{2<s<k<n} q^{-1}(3)_q\alpha_{22sk}x_sx_kx_n - q^{-1}\widehat{v_{2n2}}x_n^3,
\end{aligned}$$

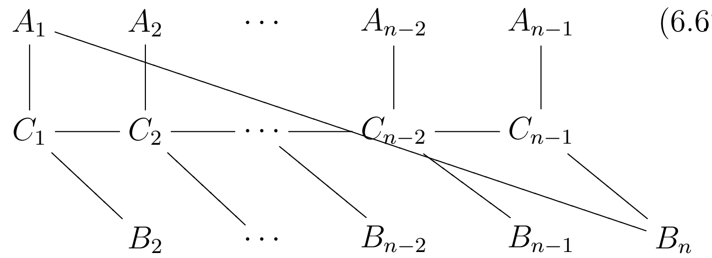
where $a_1\in\mathbb{C}\setminus\{0\}$ and $\widehat{v_{2i2}}$, α_{22kl}, β_{22}, $\widehat{v_{nn2}}$, $\alpha_{n2kn}\in\mathbb{C}$.

Let us denote the module-algebra structures of Case (1), those in Case (2), Case (3) and Case (4) in Proposition 6.1 by D, A_j, B_j and C_j respectively. For determining the module-algebra structures of $U_q(sl(3))$ on $A_q(n)$, we only need to check whether (4.58)-(4.64) hold for any $u\in\{x_1,\cdots,x_n\}$. For convenience, we introduce a notation: if the actions of k_s, e_s, f_s are of the type A_i and the actions of k_t, e_t, f_t are of the type B_j, they determine a module-algebra structure of $U_q(sl(3))$ on $A_q(n)$ for $s=1$, $t=2$ or $s=2$, $t=1$, then we say A_i and B_j are compatible. By some computations, we can obtain that D and D are compatible, A_i and B_j are compatible if and only if $i=1$ and $j=n$, A_i and C_j are compatible if and only if $i=j$, B_i and C_j are compatible if and only if $j=i+1$, C_i and C_j are compatible if and only if $i=j+1$ or $i=j-1$, and any two other cases are not compatible. As before, we use two adjacent vertices to mean two classes of module-algebra structures of $U_q(sl(3))$ on $A_q(n)$.

Therefore, by the above discussion, similar to that in Section 5, we can obtain the following proposition.

Proposition 6.3. *For $n \geq 4$, there are the module-algebra structures of $U_q(sl(3))$ on $A_q(n)$ as follows:*

$$D \longrightarrow D\,,$$



(6.67)

Here, every two adjacent vertices determine two classes of module-algebra structures of $U_q(sl(3))$ on $A_q(n)$.

Then, for determining the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$, we have to find the pairs of vertices which are not adjacent in (6.67) and satisfy the following relation: $k_ie_j(x_s) = e_jk_i(x_s)$, $k_je_i(x_s) = e_ik_j(x_s)$, $k_if_j(x_s) = f_jk_i(x_s)$, $k_jf_i(x_s) = f_ik_j(x_s)$, $e_ie_j(x_s) = e_je_i(x_s)$, $e_if_j(x_s) = f_je_i(x_s)$, $f_if_j(x_s) = f_jf_i(x_s)$ where one vertex corresponds to the actions of k_i, e_i and f_i and the other vertex corresponds to the actions of k_j, e_j and f_j, $s \in \{1, \cdots, n\}$. It is easy to check that A_i and C_j satisfy the above relations if and only if $i < j$ or $i > j+1$, B_i and C_j satisfy the above relations if and only if $i < j$ or $i > j+1$, C_i and C_j satisfy the above relations if and only if $i \neq j+1$ or $j \neq i+1$, and any other two vertices do not satisfy the above relations.

We also use m adjacent vertices to mean two classes of the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$. For example,

$$B_n \longrightarrow A_1 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_{m-2}$$

determines two classes of the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$ as follows: the one is that the actions of k_1, e_1, f_1 are of the type B_n, those of k_2, e_2, f_2 are of the type A_1 and those of k_i, e_i, f_i are of the type C_{i-2} for any $3 \leq i \leq m$. The other is that the actions of k_i, e_i, f_i are of the type C_{m-1-i} for any $1 \leq i \leq m-2$, those of k_{m-1}, e_{m-1}, f_{m-1} are of the type A_1 and those of k_m, e_m, f_m are of the type B_n.

Therefore, we obtain the following theorem.

Theorem 6.4. *For $m \geq 3$, $n \geq 4$, the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$ are as follows:*

$$\underbrace{D \longrightarrow D \longrightarrow \cdots \longrightarrow D}_{m}\,, \tag{6.68}$$

$$A_i \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_{i+m-2}\,, \tag{6.69}$$

$$C_i \longrightarrow C_{i+1} \longrightarrow \cdots \longrightarrow C_{i+m-2} \longrightarrow B_{i+m-1}\,, \tag{6.70}$$

$$C_i \longrightarrow C_{i+1} \longrightarrow \cdots \longrightarrow C_{i+m-1}\,, \tag{6.71}$$

$$A_1 \longrightarrow B_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_{n+2-m}\,, \tag{6.72}$$

where $n+2-m>1$,

$$B_n \text{ —— } A_1 \text{ —— } C_1 \text{ —— } \cdots \text{ —— } C_{m-2}\ , \tag{6.73}$$

where $m-2<n-1$.

Here, every such diagram corresponds to two classes of the module-algebra structures of $U_q(sl(m+1))$ *on* $A_q(n)$.

Remark 6.5. When $m=n-1$ and the indexes of the vertices of the Dynkin diagram are given $1,\ \cdots,\ n-1$ from the left to the right, the actions correspond to (6.71), i.e., $C_1 \text{ —— } C_2 \text{ —— } \cdots \text{ —— } C_{n-1}$ is the case discussed in [15]. In addition, we are sure that when $m \geq n+1$, all the module-algebra structures of $U_q(sl(m+1))$ on $A_q(n)$ are of the type in (6.68), since there are no paths whose length is larger than $n+1$ and any two vertices which are not adjacent in this path have no edge connecting them in (6.67). The detailed proof may be similar to that in Section 5. Moreover, the module-algebra structures of the quantum enveloping algebras corresponding to the other semisimple Lie algebras on $A_q(n)$ can be considered in the same way.

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